5 Nuclear Models

5.1 Introduction

In the previous chapters we have talked about the impossibility of obtaining the properties of a system of A nucleons starting from its constituents and their underlying interactions, and it was clearly evidenced that there is a need to use models that represent some aspects of the real problem.

The models are essentially of two classes. The first class of models assume that the nucleons interact strongly in the interior of the nucleus and that their mean free path is small. This is a situation identical to that of molecules of a liquid, and the liquid drop model belongs to this first class. These are called *collective models* and they study phenomena that involve the nucleus as a whole.

Apart from such an approach there exist a class of *independent particle models* that assume that the Pauli principle restricts the collisions of the nucleons inside the nuclear matter, leading to a larger mean free path. The several forms of shell models belong to this class.

Today we have a clear notion that the nucleus can exhibit both collective and independent particle phenomena and that each model finds its usefulness in the explanation of a specific group of nuclear properties.

5.2 The Liquid Drop Model

This model is based on the hypothesis that the nucleus has behavior identical to that of a liquid, due mainly to the fact of the occurrence, in both cases, of saturation of forces between its constituents. This idea is the starting point in obtaining an equation for the binding energy of the nucleus, introduced in chapter 4. In its simplest form this equation contains five contributory parts:

1) The main part of the binding energy is called *volume energy*. It is based on the experimental fact that the binding energy per nucleon is approximately constant (see figure 5.1);

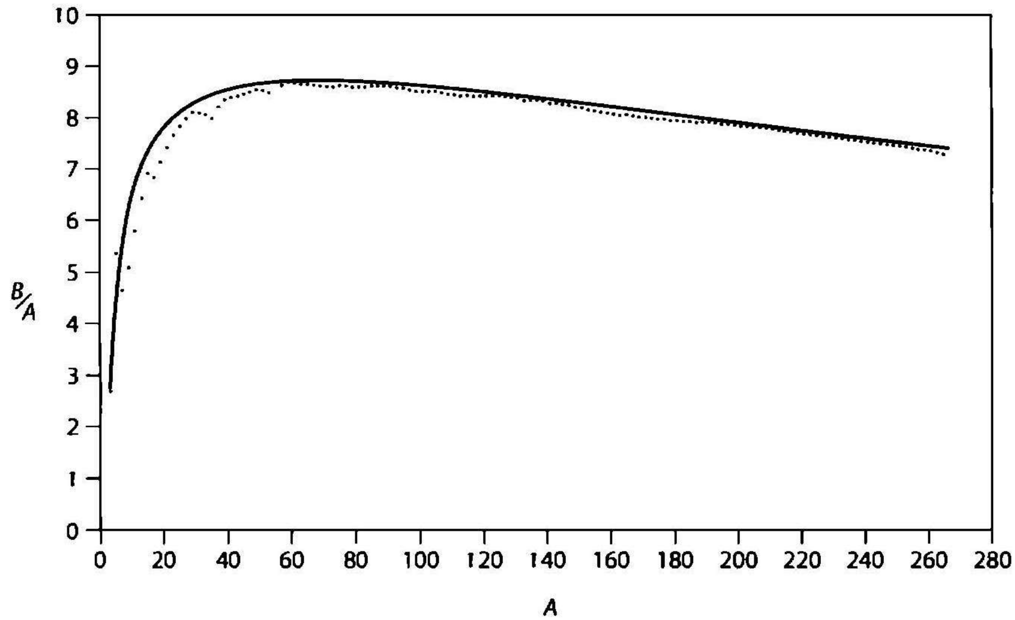

Figure 5.1 Average experimental values of B/A for A-odd nuclei, and the corresponding curve calculated by (5.7) and (5.11).

thus the total binding energy is proportional to A:

$$B_1 = a_V A. \tag{5.1}$$

If the nucleon-nucleon interaction were the same for all possible nucleon pairs, the total binding energy would be proportional to the total number of pairs, which is equal to $A(A-1)/2 \cong A^2/2$. Therefore, the binding energy per nucleon would be proportional to A. The fact that this energy is constant is due to the short range of the nuclear force, leading to the interaction of a nucleon with its neighbors. This property implies the saturation of the nuclear forces, as studied in section 4.3.

2) The surface nucleons contribute less to the binding energy since they only feel the nuclear force from the inner side of the nucleus. The number of nucleons in the surface should be proportional to the surface area, $4\pi R^2 = 4\pi r_0^2 A^{2/3}$. We should, therefore, correct (5.1) by adding the *surface energy*

$$B_2 = -a_S A^{2/3}. \tag{5.2}$$

3) The binding energy should also be smaller due to the Coulomb repulsion between the protons. The Coulomb energy of a charged sphere with homogeneous distribution and total charge Ze is given by $\frac{3}{5}(Ze)^2/R = \frac{3}{5}(e^2/r_0^2)(Z^2/A^{1/3})$. Thus, the *Coulomb energy* contributes negatively to the binding energy, given by

$$B_3 = -a_C Z^2 A^{-1/3}. \tag{5.3}$$

4) If the nucleus has a different number of protons and neutrons, its binding energy is smaller than for a symmetric nucleus. The reason for this term will be clear when we study

the Fermi gas model in the next section. This *asymmetry term* also contributes negatively and it is given by

$$B_4 = -a_A \frac{(Z - A/2)^2}{A}. \tag{5.4}$$

5) The binding energy is larger when the proton and neutron numbers are even (even-even nuclei); it is smaller when one of the numbers is odd (odd nuclei) and also when both are odd (odd-odd nuclei). Thus, we introduce a *pairing term*

$$B_5 = \begin{cases} +\delta & \text{for even-even nuclei,} \\ 0 & \text{for odd nuclei,} \\ -\delta & \text{for odd-odd nuclei.} \end{cases} \tag{5.5}$$

Empirically we find that

$$\delta \cong a_P A^{-1/2}. \tag{5.6}$$

Gathering the terms in the equations above we obtain

$$B(Z, A) = a_V A - a_S A^{2/3} - a_C Z^2 A^{-1/3} - a_A \frac{(Z - A/2)^2}{A} + \frac{(-1)^Z + (-1)^N}{2} a_P A^{-1/2}. \tag{5.7}$$

Substituting into (4.9) we obtain an equation for the mass of a nucleus,

$$\begin{aligned} m(Z, A) = {} & Z m_p + (A - Z) m_n - a_V A + a_S A^{2/3} \\ & + a_C Z^2 A^{-1/3} + a_A \frac{(Z - A/2)^2}{A} - \frac{(-1)^Z + (-1)^N}{2} a_P A^{-1/2}. \end{aligned} \tag{5.8}$$

Expression (5.8) is known as the *semi-empirical mass formula* or Weizsäcker formula [We35]. The constants appearing in (5.8) are determined empirically, that is, from the data analysis. A good adjustment is obtained using [Wa58]:

$$\begin{aligned} & a_V = 15.85\ \text{MeV}/c^2, \quad a_S = 18.34\ \text{MeV}/c^2, \quad a_C = 0.71\ \text{MeV}/c^2, \\ & a_A = 92.86\ \text{MeV}/c^2, \quad \text{and} \quad a_P = 11.46\ \text{MeV}/c^2. \end{aligned} \tag{5.9}$$

However, other good groups of parameters can also be found. Observe that a small variation in a_V or a_S leads to a large variation in the other parameters, due to a larger relevance of the corresponding terms in the mass formula.

Figure 5.1 exhibits a comparison of (5.7) to experimental data for odd nuclei. For $A < 20$ there is an absence of agreement with experience. This is expected, since light nuclei are not so similar to a liquid drop.

Equation (5.8) allows us to deduce important nuclear properties. Observe that this equation is quadratic in Z. For odd nuclei one has a parabola, as viewed in figure 5.2.

For A even, we obtain two parabolas due to the pairing energy $\pm\delta$. Nuclei with a given Z can decay into neighbors by β^+ (positron) or β^- (electron) particle emission. In figure 5.2 we see that for a nucleus with odd A there is only one stable isobar, while for even A countless stable isobars are possible.

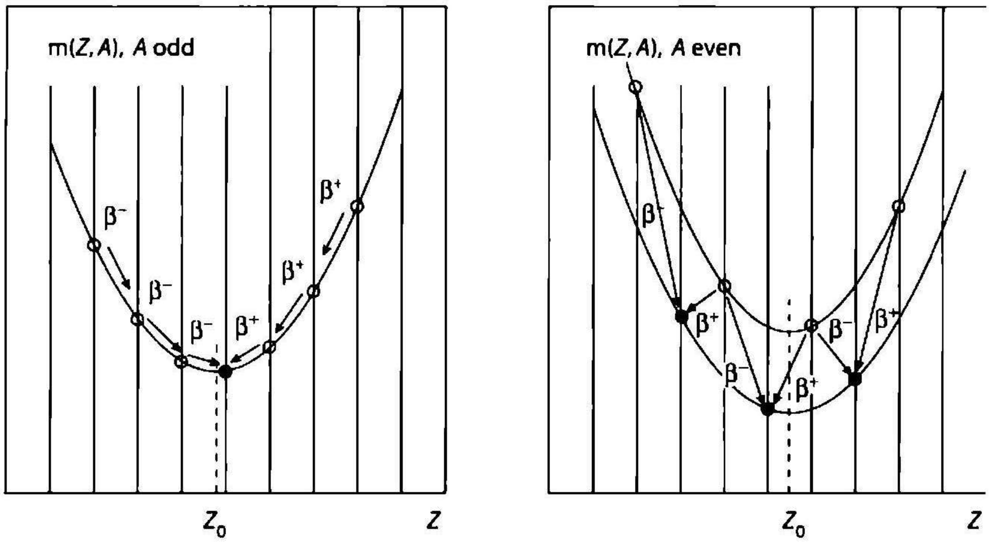

Figure 5.2 Mass of nuclei with a fixed A. The stable nuclei are represented by solid circles.

Fixing the value of A, the number of protons Z_0 for which $m(Z,A)$ is a minimum is obtained by

$$\left|\frac{\partial m(Z,A)}{\partial Z}\right|_{A=\text{const}} = 0. \tag{5.10}$$

From (5.8) we get

$$Z_0 = \frac{A}{2}\left(\frac{m_n - m_p + a_A}{a_C A^{2/3} + a_A}\right) = \frac{A}{1.98 + 0.015A^{2/3}}. \tag{5.11}$$

We see from (5.11) that the stability is obtained with $Z_0 < A/2$, that is, with a number of neutrons larger than that of protons. We know that this in fact happens, and figure 5.3 exhibits that the stability line obtained with (5.11) accompanies perfectly the valley of stable nuclei.

From (5.8) we can also ask if a given nucleus is stable against the emission of an α-particle. For this it is necessary that

$$E_\alpha = [m(Z,A) - m(Z-2, A-4) - m_\alpha]c^2 > 0, \tag{5.12}$$

situation that happens for $A \gtrsim 150$. We can also verify the possibility of a heavy nucleus decaying by fission, that is, breaking in two pieces of approximately the same size. This will be possible if

$$E_f = \left[m(Z,A) - 2m(Z/2, A/2)\right]c^2 > 0, \tag{5.13}$$

a relation valid for $A \gtrsim 90$.

The liquid drop model provides a good description of the average behavior of the binding energy with mass number, but it has nothing to say about other effects such as, for example, the existence of a magic number of nucleons. In figure 5.4 a plot is provided of the difference between the experimental separation energy of a neutron (see 4.10) and that calculated using the liquid drop model for about 2000 nuclei. It is well recognized that

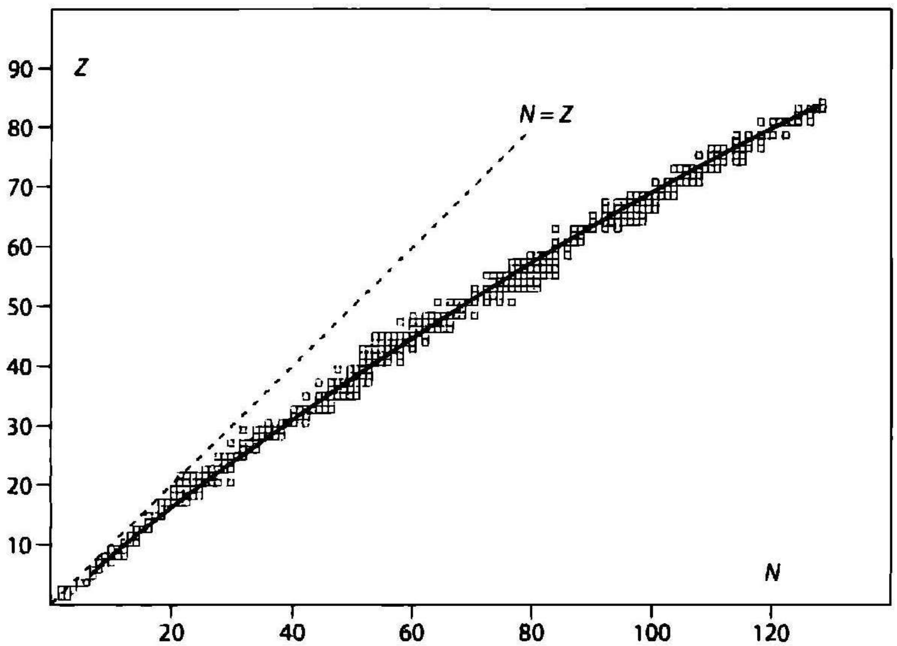

Figure 5.3 Location of the stable nuclei in the N, Z-plane. The solid line is the curve of Z_0 against $N = A - Z_0$, obtained from (5.11).

there exist values of N where the difference is positive and then drops abruptly, becoming negative. We see that these values are the magic numbers that appear in several other experiments of a different nature. The presence of these numbers is due to a structure of shells that cannot be obtained by the liquid drop model. We shall see in the following

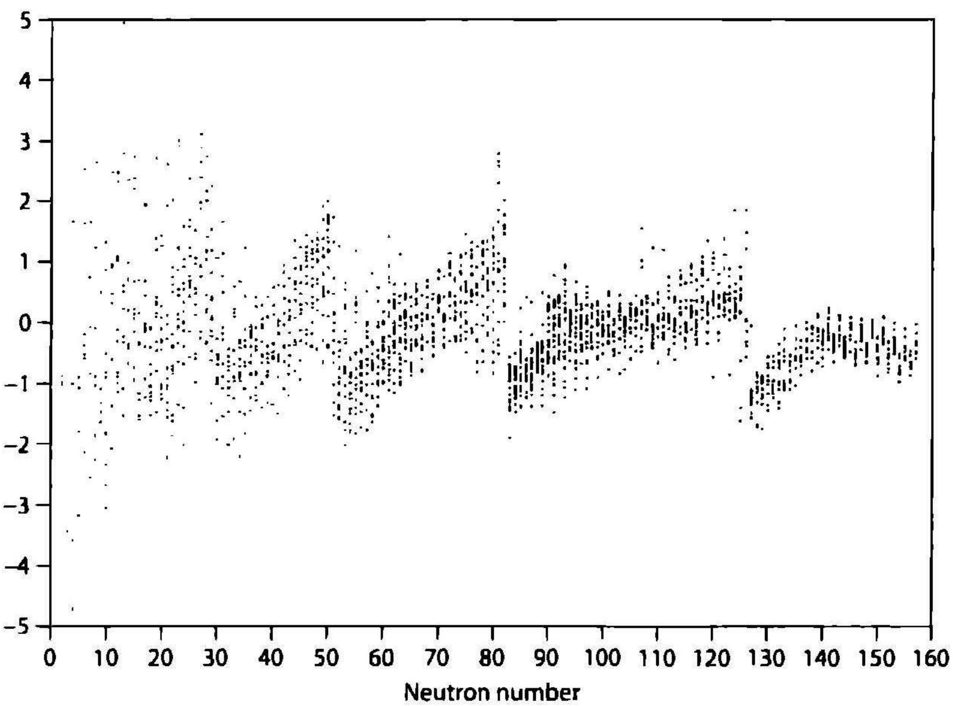

Figure 5.4 Difference (in MeV) between the experimental separation energy and the one calculated with the liquid drop model for about 2000 nuclei.

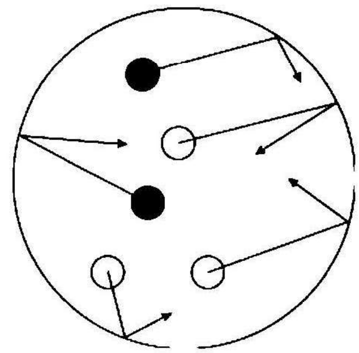

Figure 5.5 Representation of the nucleus by a Fermi gas.

sections that models that treat the nucleus as a quantum system are the only ones capable of giving a justification for the existence of these structures.

5.3 The Fermi Gas Model

This model, quite simple in its structure, is based on the fact that the nucleons move almost freely inside the nucleus due to the Pauli principle. Since two of them cannot occupy the same energy state, they do not scatter, as all possible final states that could be scattered to are already occupied by other nucleons. But when a nucleon approaches the surface and tries to fly off the nucleus, it suffers an attractive force by the nucleons that are left behind, forcing it to return toward the interior. Inside the nucleus it feels the attraction forces of all the nucleons that are around it, resulting in a net force approximately equal to zero. We can imagine the nucleus as a balloon, inside of which the nucleons move freely, but occupying states of different energy (figure 5.5).

The nucleons in the Fermi gas model obey the Schrödinger equation for a free particle,

$$-\frac{\hbar^2}{2m}\nabla^2\Psi = E\Psi, \tag{5.14}$$

where m is the nucleon mass and E its energy. To simplify let us assume that, instead of a sphere, the region to which the nucleons are limited to is the interior of a cube. The final results of our calculation will be independent of this hypothesis. In this way, Ψ will have to satisfy the boundary conditions

$$\Psi(x,y,z) = 0 \tag{5.15}$$

for

$$x = 0,\ y = 0,\ z = 0 \qquad \text{and} \qquad x = a,\ y = a,\ z = a,$$

where a is the side of the cube. The solution of (5.14) and (5.15) is given by

$$\Psi(x,y,z) = A\sin(k_x x)\sin(k_y y)\sin(k_z z), \tag{5.16}$$

since

$$k_x a = n_x\pi, \quad k_y a = n_y\pi, \quad \text{and } k_z a = n_z\pi, \tag{5.17}$$

where n_x, n_y, and n_z are positive integers and A is a normalization constant.

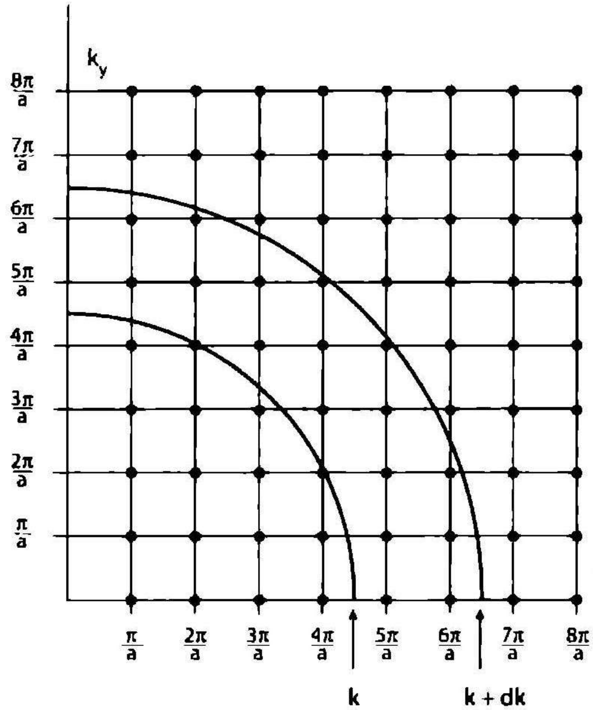

Figure 5.6 Allowed states in the part of the momentum space contained in the k_x, k_y-plane. Each state is represented by a point in the lattice.

For each group (n_x, n_y, n_z) we have an energy

$$E(n_x, n_y, n_z) = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2) = \frac{\hbar^2 \pi^2}{2ma^2} n^2, \tag{5.18}$$

where $n^2 = n_x^2 + n_y^2 + n_z^2$.

Equations (5.17) and (5.18) represent the quantization of a particle in a box, where $\mathbf{k} \equiv (k_x, k_y, k_z)$ is the momentum (divided by $\hbar$) of the particle in the box. Due to the Pauli principle, a given momentum can only be occupied by at most four nucleons: two protons with opposite spins and two neutrons with opposite spins. Consider the space of vectors $\mathbf{k}$: by virtue of (5.18), for each cube of side length π/a in this space only one point exists that represents a possible solution of the relationship (5.16). The possible number of solutions (see figure 5.6) $n(k)$ with magnitude $\mathbf{k}$ between k and $k + dk$ is given by the ratio between the volume of the spherical slice displayed in the figure and the volume $(\pi/a)^3$ for each allowed solution in the $\mathbf{k}$-space:

$$dn(k) = \frac{1}{8} 4\pi k^2 dk \frac{1}{(\pi/a)^3}, \tag{5.19}$$

where $4\pi k^2 dk$ is the volume of a spherical box in the $\mathbf{k}$-space with radius between k and $k + dk$. Only $\frac{1}{8}$ of the shell is considered, since only positive values of k_x, k_y, and k_z are necessary for counting all the states with eigenfunctions defined by (5.16). With the aid of (5.18) we can make the energy appear explicitly in (5.19):

$$dn(E) = \frac{\sqrt{2}\, m^{3/2} a^3}{2\pi^2 \hbar^3} E^{1/2} dE. \tag{5.20}$$

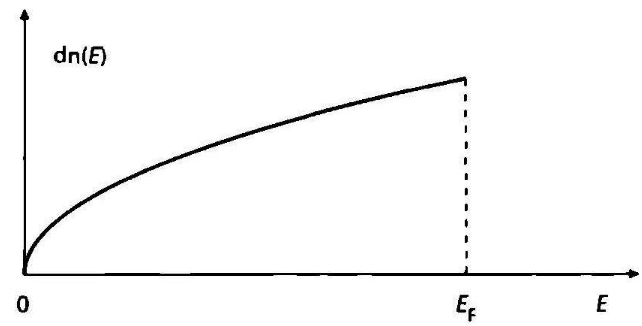

Figure 5.7 Fermi distribution for $T = 0$.

The total number of possible states of the nucleus is obtained by integrating (5.20) from 0 to the minimum value needed to include all the nucleons. This value, E_F, is called the *Fermi energy*. Thus, we obtain

$$n(E_F) = \frac{\sqrt{2}\, m^{3/2} a^3}{3\pi^2 \hbar^3} E_F^{3/2} = \frac{A}{4}, \tag{5.21}$$

where the last equality is due to the mentioned fact that a given state can be occupied by four nucleons. Inverting (5.21) we obtain

$$E_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 \rho}{2} \right)^{2/3}, \tag{5.22}$$

where $\rho = A/a^3$. We assume that the maximum energy is the same for both nucleons, which means equal values for protons and neutrons. If that is not true, the Fermi energy for protons and neutrons will be different. If $\rho_p = Z/a^3$ and $\rho_n = N/a^3$ are the respective proton and neutron densities, we will have

$$E_F(p) = \frac{\hbar^2}{2m} (3\pi^2 \rho_p)^{2/3} \tag{5.23}$$

and

$$E_F(n) = \frac{\hbar^2}{2m} (3\pi^2 \rho_n)^{2/3}, \tag{5.24}$$

for the corresponding Fermi energies.

The number of nucleons with energy between E and $E + dE$, given by (5.20), is plotted in figure 5.7 as a function of E. This distribution of particles is referred to as the *Fermi distribution* for $T = 0$, which is characterized by the absence of any particle with $E > E_F$. This corresponds to the ground state of the nucleus. An excited state ($T > 0$) can be obtained by the passage of a nucleon to a state above the Fermi level, leaving a vacancy (hole) in the energy state it previously occupied.

If we use $\rho = 1.72 \times 10^{38}$ nucleons/cm^3 = 0.172 nucleons/fm^3, which is the approximate density of all nuclei with $A \gtrsim 12$, we obtain

$$k_F = \frac{\sqrt{2mE_F}}{\hbar} = 1.36\ \text{fm}^{-1}, \tag{5.25}$$

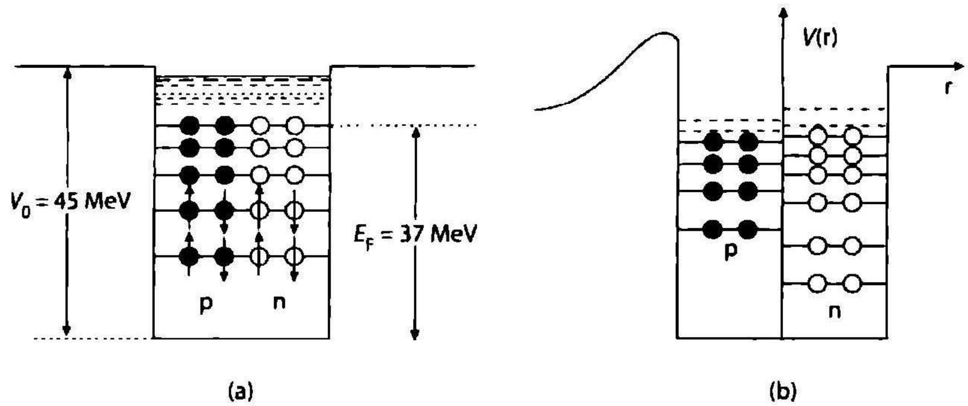

Figure 5.8 a) Potential well and states of a Fermi gas for $T = 0$. b) When one takes into account the Coulomb force. The potentials for protons and neutrons are different and we can imagine a well for each nucleon type. In two of the levels we show the spins associated to each nucleon.

which corresponds to

$$E_F = 37 \text{ MeV}. \tag{5.26}$$

We know that the separation energy of a nucleon is of the order of 8 MeV. Thus, the nucleons are not inside a well with infinite walls as we supposed, but in a well with depth $V_0 \cong (37 + 8)$ MeV = 45 MeV (figure 5.8a).

We are now prepared to explain the origin of the asymmetry term (5.4) of the liquid drop model. Let us imagine the nucleus as a mixture of a proton gas with Fermi energy $E_F(\mathrm{p})$ and a neutron gas with Fermi energy $E_F(\mathrm{n})$. Taking C as a constant, we can write

$$E_F(\mathrm{p}) = C(Z/A)^{2/3}, \quad E_F(\mathrm{n}) = C(N/A)^{2/3}. \tag{5.27}$$

If dn is the number of particles with energy between E and $E + dE$, then the total energy E_T of the gas is written as

$$E_T = \int_0^{E_F} E\, dn = \frac{3}{5} Z E_F(\mathrm{p}) \qquad \text{for protons,}$$
$$= \frac{3}{5} N E_F(\mathrm{n}) \qquad \text{for neutrons,} \tag{5.28}$$

where we used (5.20). Defining C' as a new constant, we can write the total energy as

$$E(Z, A) = C' A^{-2/3} (Z^{5/3} + N^{5/3}), \tag{5.29}$$

with the constraint $Z + N = A$. The minimum of the energy (5.29) happens when $Z = N = A/2$. Calling $E(Z, A)_{\text{min}}$ that value, let us calculate

$$\Delta E = E(Z, A) - E(Z, A)_{\text{min}} = C' A^{-2/3} \left[N^{5/3} + Z^{5/3} - 2(A/2)^{5/3} \right]. \tag{5.30}$$

Let us define $D = (N - Z)/2 = N - A/2 = A/2 - Z$; we have

$$\Delta E = C' A^{-2/3} \left[(A/2 + D)^{5/3} + (A/2 - D)^{5/3} - 2(A/2)^{5/3} \right]. \tag{5.31}$$

Expanding $(D + A/2)^{5/3}$ and $(-D + A/2)^{5/3}$ in a Taylor series

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2}f''(a) + \cdots, \tag{5.32}$$

we obtain

$$\Delta E = \frac{10}{9}C'\frac{(Z - A/2)^2}{A} + \cdots. \tag{5.33}$$

We see that the imbalance between the proton and the neutron number increases the energy of the system (decreasing the binding energy) by the amount specified in (5.33). This justifies the existence of the asymmetry term

$$B_4 = -a_A\frac{(Z - A/2)^2}{A}$$

in the liquid drop model.

Despite its simplicity, the Fermi gas model is able to explain many of the nuclear properties discussed in the previous chapter. At the beginning, the occupation of the states indicates that in a light nucleus $Z \cong N$, since in this way the energy is lowered. For heavy nuclei the Coulomb force makes the proton well shallower than that for neutrons; as a consequence, the proton number is smaller than that of neutrons, being with agreement with the practical occurrence (see figure 5.8b). Another explained characteristic is the verified abundance of even-even nuclei contrasted to the almost nonexistence of stable odd-odd nuclei. It is easy to see why this happens: when we have a nucleon isolated in a level, the lower possible state of energy for a subsequent nucleon is in that same level. In other words, in an odd-odd nucleus we have one isolated proton and one isolated neutron, each in its potential well. But between those states there is, generally, a difference of energy, creating the possibility of passage of one of the nucleons to the well of the other through β-emission and, thus, the nucleus returns to stability.

Finally, we notice that we have described here an independent particle model, and the above results predict the success applying that idea to more sophisticated models, as for instance the shell model that we will study next.

5.4 The Shell Model

This model admits that the nucleons move within the nucleus independently of each other, in the same spirit as the Fermi gas model. The difference is that now the nucleons are not treated as free particles but are subject to a central potential, similar to the central potential that acts on electrons in the atom. At first sight the idea is a bit strange because we cannot, as in the atomic case, identify the origin of a such potential. This difficulty is resolved by assuming that each nucleon moves in an average potential created by the other nucleons, a potential that should be determined in a way to best reproduce the experimental results.

The first proposals of the model appeared at the end of the 1920s, motivated by the fluctuations in the relative abundance and masses of the nuclei along the periodic table.

However, the lack of an apparent theoretical basis and the low acceptance of the idea of independent motion of nucleons, together with poor initial results, meant that the model took a long time to succeed. Finally, the introduction of a spin-orbit term, in 1949, established in a definitive way the shell model as an important tool of vast use in nuclear physics.

We are going to describe the shell model idea in a more formal way. The exact Hamiltonian for a problem of A bodies can be written as

$$H = \sum_{i}^{A} T_i(\mathbf{r}_i) + V(\mathbf{r}_1, \ldots, \mathbf{r}_A). \tag{5.34}$$

where T is the kinetic energy operator and V the potential function.

If we restrict ourselves to two-body interactions (e.g., nucleon-nucleon interaction), (5.34) takes the form

$$H = \sum_{i}^{A} T_i(\mathbf{r}_i) + \frac{1}{2} \sum_{ji} V_{ij}(\mathbf{r}_i, \mathbf{r}_j). \tag{5.35}$$

In the model proposal, the nucleon i feels not the potential $\sum_j V_{ij}$, but a central potential $U(r_i)$, that depends only on the coordinates of nucleon i. This potential can be introduced in (5.35), with the result

$$H = \sum_{i}^{A} T_i(\mathbf{r}_i) + \sum_{i}^{A} U(r_i) + H_{res}, \tag{5.36}$$

$$H_{res} = \frac{1}{2} \sum_{ji} V_{ij}(\mathbf{r}_i, \mathbf{r}_j) - \sum_{i} U(r_i). \tag{5.37}$$

H_{res} refers to the *residual interactions*, that is, the part of potential V not embraced by the central potential U. The hope of the shell model is that the contribution of H_{res} is small or, alternatively, that the *shell model Hamiltonian*,

$$H_0 = \sum_{i=1}^{A} \left[T_i(r_i) + U(r_i) \right], \tag{5.38}$$

represents a good approximation for the exact expression of H. Later we shall see that part of the lost accuracy when we pass from (5.35) to (5.38) can be recovered by an approximated treatment of the effect of the residual interaction H_{res}.

The solutions $\Psi_1(\mathbf{r}_1), \Psi_2(\mathbf{r}_2), \ldots$, of the equation

$$H_0 \Psi = E \Psi, \tag{5.39}$$

with respective eigenvalues $E_1, E_2, \ldots$, are called *orbits* or *orbitals*. In the shell model prescription the A nucleons fill the orbitals of lower energy in a way compatible with the Pauli principle. Thus, if the sub-index 1 of Ψ_1, which represents the group of quantum numbers of the orbital 1, includes spin and isospin, we can say that the first nucleon is described by $\Psi_1(\mathbf{r}_1), \ldots,$ and the A-th by $\Psi_A(\mathbf{r}_A)$. Thus, the wavefunction

$$\Psi = \Psi_1(\mathbf{r}_1)\Psi_2(\mathbf{r}_2)\ldots\Psi_A(\mathbf{r}_A) \tag{5.40}$$

is a solution of (5.39) with eigenvalues

$$E = E_1 + E_2 + \cdots + E_A, \tag{5.41}$$

and it would be, in principle, the wavefunction of the nucleus, with energy E given by the shell model. We should have in mind, however, that we are treating a fermion system and that the total wavefunction should be antisymmetric for an exchange of coordinates of two nucleons. Such a wavefunction is obtained from (5.40) for the construction of the *Slater determinant*

$$\Psi = \frac{1}{\sqrt{A!}} \begin{vmatrix} \Psi_1(\mathbf{r}_1) & \Psi_1(\mathbf{r}_2) & & \Psi_1(\mathbf{r}_A) \\ \Psi_2(\mathbf{r}_1) & \Psi_2(\mathbf{r}_2) & & \Psi_2(\mathbf{r}_A) \\ \vdots & \vdots & & \vdots \\ \Psi_A(\mathbf{r}_1) & \Psi_A(\mathbf{r}_2) & & \Psi_A(\mathbf{r}_A) \end{vmatrix}, \tag{5.42}$$

where the change of coordinates (or of the quantum numbers) of two nucleons changes the sign of the determinant.

An inconvenience of the construction in (5.42) is that the function Ψ, by mixing well-defined angular momenta J and isotopic spin T, is no longer an eigenfunction of these operators. The solution to this difficulty involves the construction of linear combinations of Slater determinants that are eigenfunctions of J and T. The problem has a well-known solution, but it involves a great amount of calculation. It is important to make clear, however, that many of the properties of the nuclear states can be extracted from the shell model without knowledge of the wavefunction, as we will see next.

We will analyze what is obtained when one starts with potentials $U(r)$ (5.38) with well-known solutions. Let us initially examine the simple harmonic oscillator. Being a potential that always grows with distance, at first it would seem not to be adaptable to representation of the nuclear potential, which goes to zero when the nucleon is at a larger distance than the radius of the nucleus. It is expected, however, that this is not very important when we analyze just the bound states of the nucleus. The oscillator potential has the form

$$V(r) = \frac{1}{2}m\omega^2 r^2, \tag{5.43}$$

where the frequency ω should be adapted to the mass number A.

We will seek solutions of (5.43) of the type

$$\Psi(\mathbf{r}) = \frac{u(r)}{r} Y_l^m(\theta, \phi), \tag{5.44}$$

where the substitution of Ψ in the Schrödinger equation for a particle reduces the solution of (5.44) to the solution of an equation for u:

$$\frac{d^2u}{dr^2} + \left\{ \frac{2m}{\hbar^2}\left[E - V(r)\right] - \frac{l(l+1)}{r^2} \right\} u = 0. \tag{5.45}$$

The solution of (5.45) with the potential (5.43) is

$$u_{nl}(r) = N_{nl} \exp\left(-\frac{1}{2}\nu r^2\right) r^{l+1} \mathcal{V}_{nl}(r), \tag{5.46}$$

where $\nu = m\omega\hbar$ and $\mathcal{V}_{nl}(r)$ is the associated Laguerre polynomial

$$\mathcal{V}_{nl}(r) = L^{l+\frac{1}{2}}_{n+l-\frac{1}{2}}(\nu r^2) = \sum_{k=0}^{n-1} (-1)^k 2^k \begin{pmatrix} n-1 \\ k \end{pmatrix} \frac{(2l+1)!!}{(2l+2k+1)!!} (\nu r^2)^k, \tag{5.47}$$

where $L^{\alpha}_{k}(t)$ are solutions of the equation

$$t\frac{d^2L}{dt^2} + (\alpha + 1 - t)\frac{dL}{dt} + kL = 0. \tag{5.48}$$

From the normalization condition

$$\int_0^{\infty} u^2_{nl}(r)dr = 1 \tag{5.49}$$

we obtain that

$$N^2_{nl} = \frac{2^{l-n+3}(2l+2n-1)!!}{\sqrt{\pi}(n-1)![(2l+1)!!]^2} \nu^{l+\frac{3}{2}}. \tag{5.50}$$

The energy eigenvalues corresponding to the wavefunction $\Psi_{nlm}(\mathbf{r})$ are

$$E_{nl} = \hbar\omega\left(2n + l - \frac{1}{2}\right) = \hbar\omega\left(\Lambda + \frac{3}{2}\right) = E_{\Lambda}, \tag{5.51}$$

where

$$n = 1, 2, 3, \ldots, \quad l = 0, 1, 2, \ldots, \quad \text{and} \quad \Lambda = 2n + l - 2. \tag{5.52}$$

For each value of l there are $2(2l+1)$ states with the same energy (degenerate states). The factor 2 is due to two spin states. However, the eigenvalues that correspond to the same value of $2n + l$ (same value of Λ) are also degenerate. As $2n = \Lambda - l + 2 =$ even, a given value of Λ corresponds to the degenerate eigenstates

$$(n, l) = \left(\frac{\Lambda+2}{2}, 0\right)\left(\frac{\Lambda}{2}, 2\right), \ldots, (2, \Lambda - 2), (1, \Lambda) \tag{5.53}$$

for Λ even and

$$(n, l) = \left(\frac{\Lambda+1}{2}, 1\right)\left(\frac{\Lambda-1}{2}, 3\right), \ldots, (2, \Lambda - 2), (1, \Lambda) \tag{5.54}$$

for Λ odd.

We obtain then that the neutron or proton numbers with eigenvalues E_{Λ} are given by (we will use $l = 2k$ or $2k+1$, in the case that Λ is even or odd)

$$N_{\Lambda} = \sum_{k=0}^{\Lambda/2} 2[2(2k) + 1] \qquad \text{for } \Lambda \text{ even}, \tag{5.55}$$

$$N_{\Lambda} = \sum_{k=0}^{\Lambda-1/2} 2[2(2k+1) + 1] \qquad \text{for } \Lambda \text{ odd}. \tag{5.56}$$

In both cases, the result is

$$N_\Lambda = (\Lambda + 1)(\Lambda + 2). \tag{5.57}$$

The quantum number Λ defines a *shell* and each shell can accommodate N_Λ protons and N_Λ neutrons.

The accumulated number of particles for all the levels up to Λ is

$$\sum_\Lambda N_\Lambda = \frac{1}{3}(\Lambda + 1)(\Lambda + 2)(\Lambda + 3). \tag{5.58}$$

Based on these results, we can make an estimate of the frequency ω of the harmonic oscillator applied to a nucleus with atomic number A. For a harmonic oscillator, the expectation value of the kinetic energy of a given state is equal to the expectation value of the potential energy. Thus, the sum of the energies of the occupied states in a nucleus of mass A is

$$E = m\omega^2 A \langle r^2 \rangle. \tag{5.59}$$

We can estimate $\langle r^2 \rangle$ using

$$\langle r^2 \rangle \cong \frac{3}{5} R^2, \tag{5.60}$$

with $R \cong 1.2A^{\frac{1}{3}}$. Assuming that $N = Z$ and that all states up to an energy E_Λ are occupied, one gets

$$A = \sum_{\Lambda=0}^{\Lambda_0} 2N_\Lambda = \frac{2}{3}(\Lambda_0 + 1)(\Lambda_0 + 2)(\Lambda_0 + 3) \cong \frac{2}{3}(\Lambda_0 + 2)^2 + \text{terms of order}(\Lambda_0) \tag{5.61}$$

and

$$\frac{E}{\hbar\omega} = \sum_{\Lambda=0}^{\Lambda_0} 2N_\Lambda \left(\Lambda + \frac{3}{2}\right) \cong \frac{1}{2}(\Lambda_0 + 2)^4 - \frac{1}{3}(\Lambda_0 + 2)^3 + \cdots . \tag{5.62}$$

Eliminating $\Lambda_0 + 2$ from the equations above and keeping terms of larger order in $\Lambda_0 + 2$, one gets

$$\frac{E}{\hbar\omega} \cong \frac{1}{2}\left(\frac{3}{2}A\right)^{\frac{4}{3}}. \tag{5.63}$$

Using (5.59) and (5.60) gives

$$\hbar\omega \cong 41A^{-\frac{1}{3}} \text{ MeV}. \tag{5.64}$$

The giant dipole resonances are excitations with $\Delta l = \pm 1$. The position of the peak varies with the mass of the nucleus as $A^{-\frac{1}{3}}$, being a good example of application of (5.64).

The levels predicted by the harmonic oscillator are given in table 5.1. We can observe that the closed shells appear in levels 2, 8, and 20, in agreement with the experimental facts, since the nuclei should close their shells (of protons and of neutrons) with a magic number. But, the same does not happen in closed shells for nucleon numbers larger than 20, which is in disagreement with experience.

Table 5.1 Nucleon distribution for the first shells of a simple harmonic oscillator. The last column indicates the total number of neutrons (or protons) accumulated up to that shell.

$\Lambda = 2n + l - 2$	$E/\hbar\omega$	l	States	N_Λ = number of neutrons (protons)	Total
0	3/2	0	1s	2	2
1	5/2	1	1p	6	8
2	7/2	0,2	2s,1d	12	20
3	9/2	1,3	2p,1f	20	40
4	11/2	0,2,4	3s,2d,1g	30	70
5	13/2	1,3,5	3p,2f,1h	42	112
6	15/2	0,2,4,6	4s,3d,2g,1i	56	168

For an infinite square well we have the same approximate situation. The solutions for that potential obey the equation

$$\frac{d^2u}{dr^2} + \left[\frac{2m}{\hbar^2}E - \frac{l(l+1)}{r^2}\right]u = 0, \tag{5.65}$$

whose solutions

$$u = A\, r\, j_l(kr) \tag{5.66}$$

involve spherical Bessel functions that obey the boundary condition

$$j_l(kR) = 0, \tag{5.67}$$

where R is the radius of the nucleus and $k = \sqrt{2mE}/\hbar$. From (5.66) and (5.67) we build the allowed states for that potential. Table 5.2 shows these states, the nucleon numbers admitted in each of them, and the nucleon numbers that close the shells. Again here the magic numbers are only reproduced in the initial shells.

The nuclear potential should actually have an intermediary form between the harmonic oscillator and the square well, not being as smooth as the first or as abrupt as the second. It is common to use the "Woods-Saxon" form $V = V_0/\{1 + \exp[(r - R)/a]\}$, where V_0, r, and a are adjustable parameters. A numerical solution of the Schrödinger equation with such a potential does not supply us, however, with the expected results.

A considerable improvement was obtained in 1949 by Maria Mayer [Ma49] and, independently, by Haxel, Jensen, and Suess [Ha49], with the introduction of a term of spin-orbit interaction in the form

$$f(r)\mathbf{l} \cdot \mathbf{s}, \tag{5.68}$$

Table 5.2 Proton (or neutron) distribution for the first shells of an infinite square well. The principal quantum number n indicates the order in that a zero appears for a given l in eq. (5.67). Notice that here there is no longer the degeneracy in l. The third column gives the proton and neutron numbers that can be fitted in each orbit.

orbit: nl	kR	$2(2l+1)$	Total
1s	3.142	2	2
1p	4.493	6	8
1d	5.763	10	18
2s	6.283	2	20
1f	6.988	14	34
2p	7.725	6	40
1g	8.183	18	58
2d	9.095	10	68
3s	9.425	2	70

where $f(r)$ is a radial function that should be obtained by comparison with experiments. However, we will soon see that its form is not important for the effect that we want.

A spin-orbit term already appears in atomic physics as a result of the interaction between the magnetic moment of the electrons and the magnetic field created by orbital motion. In nuclear physics this term has a different nature and is related to the quantum field properties of an assembly of nucleons.

We will see that the addition of such a term to the potential of (5.43) alters the energy values. The new values are given to first order by

$$E = \int \Psi^* H \Psi = \left(n + \frac{3}{2}\right)\hbar\omega + \alpha \int \Psi^* f(r) \mathbf{l} \cdot \mathbf{s}\, \Psi, \tag{5.69}$$

where α is a proportionality constant. If we suppose now that the spin-orbit term is small and that it can be treated as a small perturbation, the wavefunctions in (5.69) are basically those of a central potential. Recalling that $\mathbf{l} \cdot \mathbf{s} = (j^2 - l^2 - s^2)/2$, we have

$$\begin{aligned} &\int \Psi^* \mathbf{l} \cdot \mathbf{s}\, \Psi = \frac{l}{2} \qquad \text{for } j = l + \frac{1}{2}, \\ &\int \Psi^* \mathbf{l} \cdot \mathbf{s}\, \Psi = -\frac{1}{2}(l+1) \qquad \text{for } j = l - \frac{1}{2}. \end{aligned} \tag{5.70}$$

Thus, the spin-orbit interaction removes the degeneracy in j and, anticipating that the best experimental result will be obtained if the orbitals for larger j have the energy lowered, we admit a negative value for α, allowing us to write for the energy increment

$$\Delta E\,|_{j=l+1/2} = -\,|\,\alpha\,|\,\langle f(r)\rangle\,\frac{l}{2}, \tag{5.71}$$

$$\Delta E\,|_{j=l-1/2} = +\,|\,\alpha\,|\,\langle f(r)\rangle\,\frac{1}{2}(l+1). \tag{5.72}$$

Figure 5.9 exhibits the level scheme of a central potential with the introduction of the spin-orbit interaction. It is easy to see the effect of (5.71) and (5.72) in the energy distribution of the levels. Referring now to a shell as a group of levels of closed energy, not necessarily associated to only one principal quantum number of the oscillator, we obtain a perfect description of all the magic numbers.

We will use this level scheme[1] to establish what one refers to as the *single particle model* or *extreme shell model*. In this version, the model allows that an odd nucleus is composed of an inert even-even core plus an unpaired nucleon, and that this last nucleon determines the properties of the nucleus. This idea was discussed earlier; what we can now do is to determine, starting from figure 5.9, in which state we find the unpaired nucleon.

Let us take ^{17}O as an example. This nucleus has a shell closed with 8 protons but has a remaining neutron above the closed core of 8 neutrons. A quick examination of figure 5.9 indicates that this neutron finds itself in level $1d_{5/2}$. We can say that the neutron *configuration* of this nucleus is

$$(1s_{1/2})^2\,(1p_{3/2})^4\,(1p_{1/2})^2\,(1s_{1/2})^2\,(1d_{5/2})^1\,,$$

it being evident that in this definition we list the filled levels for the neutrons, with the upper indices, equal to $2j+1$, indicating the number of particles in each of them. It is also common to restrict the configuration to the partially filled levels, the subshells being completely ignored. Thus, the configuration of neutrons in ^{17}O would be $(1d_{5/2})^1$. The prediction of the model is, in this case, that the spin of the ground state of ^{17}O is $\frac{5}{2}$ and the parity is positive ($l = 2$). This prediction is in agreement with experiments. For similar reasons, the model predicts that the ground state of ^{17}F is a $\frac{5}{2}^+$ state and this is indeed the measured value.

The extreme shell model works well when we have a nucleon above a closed shell, as in the examples above. It also works well for a hole (absence of a nucleon) in a closed shell. Examples of this case are the nuclei ^{15}O and ^{15}N for which the model predicts correctly a $\frac{1}{2}^-$ ground state. There are situations, however, in which the model needs a certain adaptation. Such is the case, for example, for the stable nuclei ^{203}Tl and ^{205}Tl. They have 81 protons, with a resulting hole in $\frac{11}{2}$. Their ground state is, however, $\frac{1}{2}^+$ instead of $\frac{11}{2}^-$. In order to understand what happens it is necessary to recall that the model, in the simple form we are using, totally neglects the individual interactions of the nucleons; a correction of the model would be to take into account certain nucleon-nucleon interactions that we know are present and that are part of the residual interaction (5.37).

[1] Each level of figure 5.9, characterized by the quantum numbers n, l, j, contains $2j+1$ nucleons of a same type and is also referred to as a subshell.

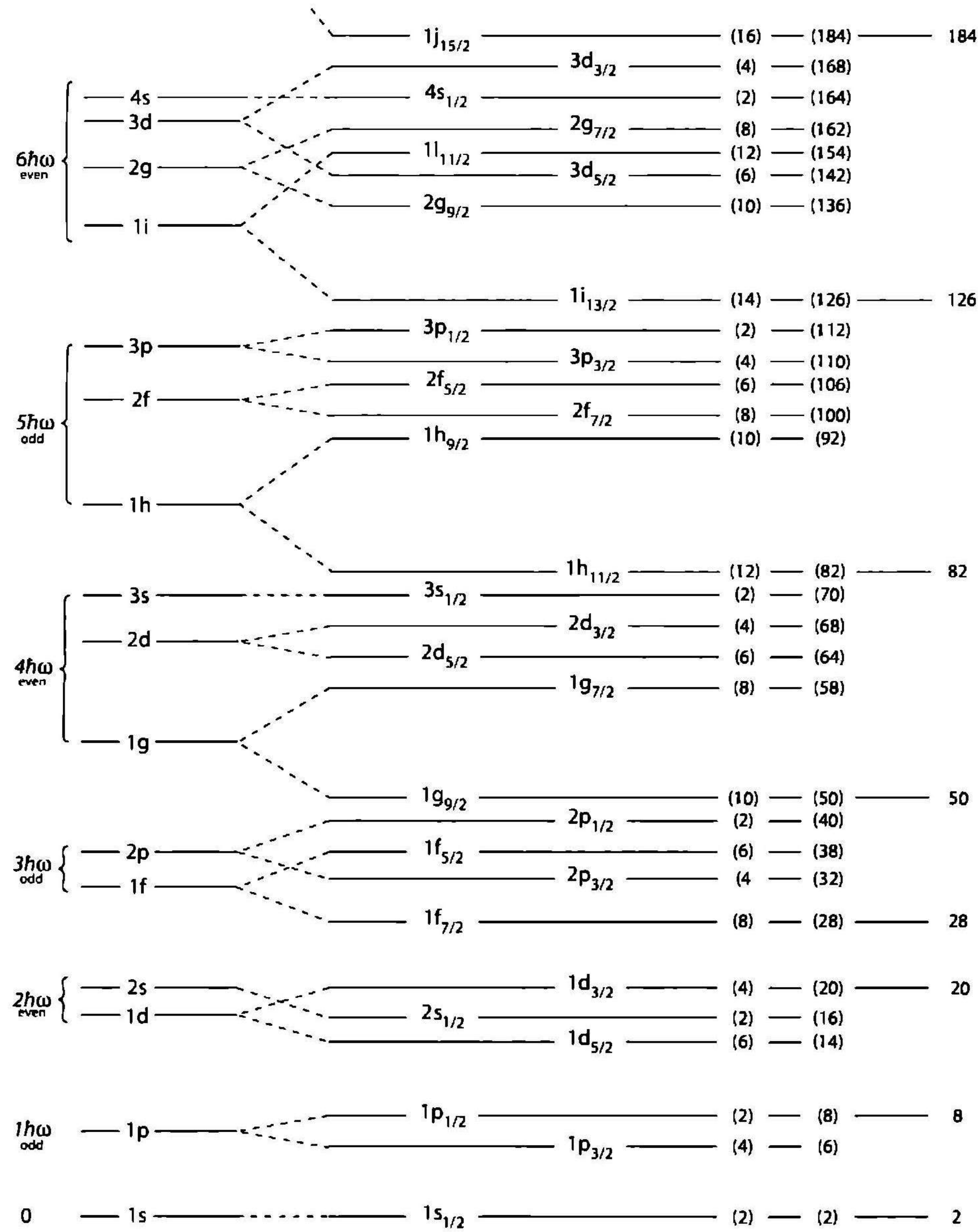

Figure 5.9 Level scheme of the shell model showing the break of the degeneracy in j caused by the spin-orbit interaction term and the emergence of the magic numbers in the shell closing. The values in the first set of parentheses indicate the number of nucleons of each type that the level admits and the values in the second set of parentheses provide the total number of nucleons of each type up to that level. Finally, the numbers outside parentheses indicate the total number of nucleons at shell-closure, reobtaining the magic numbers in their entirety. The ordering of the levels is not rigid, and there could be level inversions when changes occur in the form of the potential [MJ55].

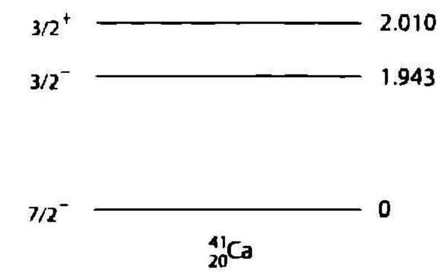

Figure 5.10 Level scheme for ^{41}Ca. The values to the right are the energies in MeV.

A class of interactions of special interest is the one that involves a proton (or a neutron) pair of equal orbits n, l, and j with symmetric values of m_j. A collision of these particles can take the pair to other orbits with the same quantum numbers n, l, j but with new projections m_j' and $-m_j'$. These collisions conserve energy (the $2j+1$ states are degenerate), angular momentum, and parity, and we expect that there is a permanent alternation between the several possible values of the pair m_j, $-m_j$. The interaction between two nucleons in these circumstances is commonly called the *pairing force*. It leads to an increase of the binding energy of the nucleus; since the nucleons belong to the same orbit, their wavefunctions have the same space distribution and the average proximity between them is maximum. As the nuclear force is attractive, this leads to an increase in the binding energy. The pairing force is responsible for the pairing term (5.6) of the mass formula (5.7). It is the same type of force that, acting between the conduction electrons of a metal, in special circumstances and at low temperatures, yields the superconductivity phenomenon [Ba57].

The pairing force increases with the value of j, since the larger the angular momentum, the larger the location of the wavefunction of the nucleon around a classical orbit, and also the stronger the argument of the previous paragraph. This implies that it is sometimes more energetically advantageous that the isolated nucleon is not in the last level but below it, leaving the last level for a group of paired nucleons. This happens in our example of Tl; the hole is located not in $h_{\frac{11}{2}}$ but in $s_{\frac{1}{2}}$, leaving the orbital $h_{\frac{11}{2}}$ (of high j) occupied by a pair of protons. Another example is ^{207}Pb, for which the hole in the closed shell of 126 neutrons is not in $i_{\frac{13}{2}}$ but in $p_{\frac{1}{2}}$, resulting in the value $\frac{1}{2}^-$ for the spin of its ground state.

An example of another kind is ^{23}Na. This nucleus has the last 3 protons in the orbital $1d_{\frac{5}{2}}$. The value of its spin is, however, $\frac{3}{2}$. This is an example of a flaw in the predictions of the extreme shell model. Here, it is the coupling between the three nucleons that determines the value of the spin and not separately the value of j of each of them. This type of behavior will be analyzed in the following section.

Having established the outline of operation of the shell model, it is easy to apply it to the determination of the excited states of nuclei. In the case of ^{41}Ca (figure 5.10), the ground state is $\frac{7}{2}^-$, since the extra neutron occupies the orbital $f_{\frac{7}{2}}$. The first excited level corresponds to a jump of that neutron to $p_{\frac{3}{2}}$, generating the state $\frac{3}{2}^-$. The second excited

Table 5.3 Determination of the spin of odd-odd nuclei. The 7th column lists the possible values of j predicted by the Nordheim rules and the last column lists the experimental values with their respective parities.

Nucleus	Proton Z	State	Neutron N	State	$\mathcal{N}$	j_{pred}	j_{exp}
^{14}N	7	$p_{\frac{1}{2}}$	7	$p_{\frac{1}{2}}$	−1	0 or 1	1^+
^{42}K	19	$d_{\frac{3}{2}}$	23	$f_{\frac{7}{2}}$	0	2	2^-
^{80}Br	35	$p_{\frac{3}{2}}$	45	$p_{\frac{1}{2}}$	0	1	1^+
^{208}Tl	81	$s_{\frac{1}{2}}$	127	$g_{\frac{9}{2}}$	1	4 or 5	5^+

state, $\frac{3}{2}^+$, is obtained by the passage of a neutron from $1d_{\frac{3}{2}}$ to $1f_{\frac{7}{2}}$, leaving a hole in $1d_{\frac{3}{2}}$. It should be noticed, however, that in a very few cases the single particle model gets to a reasonable prediction of spins and energy of excited states. For such a purpose it is necessary to have a more sophisticated version of the shell model, to be described in the following section.

As a last topic in this section we will examine the situation of one odd-odd nucleus. In this case, two nucleons, a proton and a neutron, are unpaired. If $\mathbf{j}_p$ and $\mathbf{j}_n$ are the respective angular momenta of the nucleons, the angular momentum of the nucleus j can have values from $|j_p - j_n|$ to $j_p + j_n$. L. W. Nordheim proposed, in 1950 [No50], rules to determine the most probable value of the spin of an odd-odd nucleus. Defining the *Nordheim number*

$$\mathcal{N} = j_p - l_p + j_n - l_n, \tag{5.73}$$

those rules establish that

a) If $\mathcal{N} = 0, j = |j_p - j_n|$; this is called the *strong rule*.
b) If $\mathcal{N} = \pm 1, j = j_p + j_n$ or $j = |j_p - j_n|$, the *weak rule*.

The Nordheim rules reveal a tendency (not widespread, because there are exceptions) to an alignment of the intrinsic spins, as in the state $j = 1$ of the deuteron. Table 5.3 shows some practical examples.

When we have an incomplete subshell, the states that it can form with k nucleons is degenerate. The presence of residual forces among those nucleons separates the states in energy; that is, the degeneracy is removed. The angular momentum of each state is one of the possible values that result from adding k angular momenta $\mathbf{j}$. The ground state will be the lowest energy of the group, and its angular momentum will not necessarily be equal to j^2.

Table 5.4 Possible values of the projection of the total angular momentum when two identical nucleons occupy the level $j = \frac{5}{2}$.

m_1	m_2	m	m_1	m_2	m
$\frac{5}{2}$	$\frac{3}{2}$	4	$\frac{3}{2}$	$-\frac{5}{2}$	−1
$\frac{5}{2}$	$\frac{1}{2}$	3	$\frac{1}{2}$	$-\frac{1}{2}$	0
$\frac{5}{2}$	$-\frac{1}{2}$	2	$\frac{1}{2}$	$-\frac{3}{2}$	−1
$\frac{5}{2}$	$-\frac{3}{2}$	1	$\frac{1}{2}$	$-\frac{5}{2}$	−2
$\frac{5}{2}$	$-\frac{5}{2}$	0	$-\frac{1}{2}$	$-\frac{3}{2}$	−2
$\frac{3}{2}$	$\frac{1}{2}$	2	$-\frac{1}{2}$	$-\frac{5}{2}$	−3
$\frac{3}{2}$	$-\frac{1}{2}$	1	$-\frac{3}{2}$	$-\frac{5}{2}$	−4
$\frac{3}{2}$	$-\frac{3}{2}$	0			

Before we examine the possible values of the angular momentum, it is convenient to discuss the composition of the valence level. When we filled out the levels of the shell model we placed $2j+1$ protons in a level n, l, j, and also $2j+1$ neutrons, since the Pauli principle does not restrict the presence of different nucleons in the same quantum state. We can, in this way, think of a "proton" well and a "neutron" well, filled in independent ways. When we have a heavy nucleus, the Coulomb force implies that the proton well is shallower than the neutron well. We have a situation similar to the one shown in figure 5.8b for the case of the Fermi gas. Thus, the shell where we find the last protons and the shell where we find the last neutrons, both close to the Fermi level, can be different, corresponding to quite different wavefunctions and of small space overlap. It is expected that a residual proton-neutron interaction is not important in this case. For light nuclei, on the other hand, the Coulomb effect is small and the inclusion of particle-particle interactions should treat protons and neutrons within a single context.

We will analyze the first case initially and imagine that k "last" nucleons, say, protons, reside in a subshell defined by n, l, j. The configuration for the proton well has the form

$$(n_1 l_1 j_1)^{2j_1+1} (n_2 l_2 j_2)^{2j_2+2} \dots (n\, l\, j)^k.$$

If we do not take into consideration the interaction forces, the $2j+1$ states that compose the last level, where we find k valence protons, are degenerate. The presence of the interaction removes the degeneracy; thus, for example, if we have two protons ($k=2$) in a level $j=\frac{5}{2}$, we can follow the picture shown in table 5.4. The first two columns list the possible values of m_j for the two particles allowed by the Pauli principle, and the last column lists the value $m_j = m_j(1) + m_j(2)$. We know that the possible values of the resulting angular momentum are located in the range $|j_1 - j_2| < j < j_1 + j_2$, that is, j could, in this example,

Table 5.5 Possible values for the total angular momentum of k identical nucleons placed in a sub-shell of angular moment j. The numbers in parentheses indicate the number of times the value repeats.

j	k	
$\frac{1}{2}$	1	$\frac{1}{2}$
$\frac{3}{2}$	1	$\frac{3}{2}$
	2	0,2
$\frac{5}{2}$	1	$\frac{5}{2}$
	2	0,2,4
	3	$\frac{3}{2}, \frac{5}{2}, \frac{9}{2}$
$\frac{7}{2}$	1	$\frac{7}{2}$
	2	0,2,4,6
	3	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{15}{2}$
	4	0,2(2),4(2),5,6,8
$\frac{9}{2}$	1	$\frac{9}{2}$
	2	0,2,4,6,8
	3	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}(2), \frac{11}{2}, \frac{15}{2}, \frac{17}{2}, \frac{21}{2}$
	4	0(2),2(2),3,4(3),5,6(3),7,8(2),9,10,12
	5	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}(2), \frac{7}{2}(2), \frac{9}{2}(3), \frac{11}{2}(2), \frac{13}{2}(2), \frac{15}{2}(2), \frac{17}{2}(2), \frac{19}{2}, \frac{21}{2}, \frac{25}{2}$

take the values 0, 1, 2, 3, 4, or 5. If, however, we accept the rule that each value of m_j in table 5.4 is the projection of a single value of j, we will see that only the momenta $j = 0$, 2, and 4 can exist.

When we have more than two particles in a level, the situation can become much more complex. The practical method of finding the possible values of the resulting angular momentum shown in table 5.4 can always be used, but it is quite a difficult matter when we have many nucleons. Table 5.5 shows the possibilities for several configurations, where we see new restrictions imposed by the antisymmetrization, or, equivalently, by the Pauli principle, for given values of the total angular momentum J.

If we have k nucleons in an orbital around a light core of $Z = N$, we can form $k + 1$ nuclei by varying the neutron and proton numbers in the level. The value of T for each nucleus is located inside the limits

$$|T_z| \leq T \leq \frac{k}{2}. \tag{5.74}$$

Let us take as an example the magic core $Z = N = 4$ above which two nucleons are placed. We can form the nuclei $^{10}_{4}$Be, $^{10}_{5}$B, and $^{10}_{6}$C. As $k = 2$, T can assume the values 0 and 1. The component T_z for each nucleus has the values -1, 0, and $+1$, respectively. Figure 5.11 exhibits the level diagrams of each of these nuclei. We can see that states of T equal to 1 form a triplet with a representative in each of the nuclei and states of $T = 0$

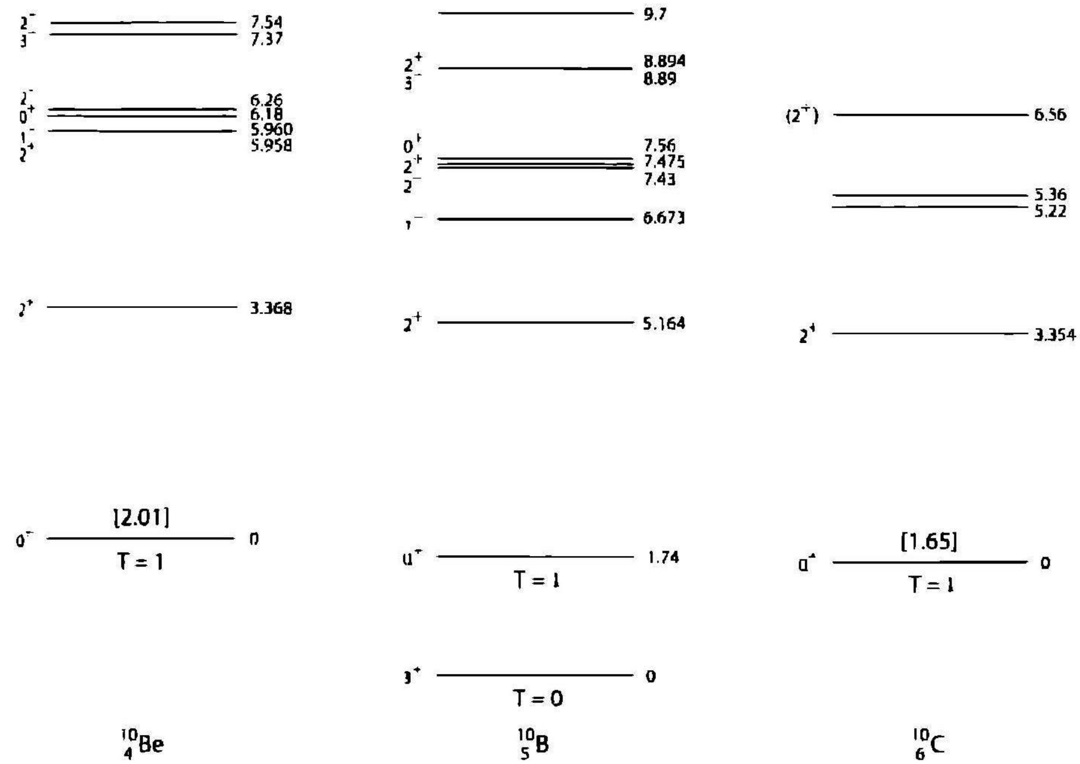

Figure 5.11 States in T = 1 of ${}^{10}_{4}$Be, ${}^{10}_{5}$B, and ${}^{10}_{6}$C, above the ground state of ${}^{10}_{5}$B with T = 0. The effect of the Coulomb energy 5.3 is cancelled by adding its values to the binding energy of each isobar. The energy values are given in MeV [Wa93].

exist only in ^{10}B, of component $T_z = 0$. The states that compose each triplet have the same approximate energy in the three nuclei; they are referred to as *isobaric analog states.* As the effect of the Coulomb energy was subtracted, this result is another indication that the nuclear forces are independent of charge. In other words, if the p-p, n-n, and p-n nuclear forces are identical, systems of A nucleons should have the same properties and, in particular, if we discount the Coulomb force, the same energy levels. The existence of states of $T = 0$ for ^{10}B that do not have partners in ^{10}Be and ^{10}C can be understood by the Pauli principle: ^{10}B has five n-p pairs while ^{10}Be and ^{10}C have four n-p pairs plus a pair of identical nucleons. Thus, the last pair of ^{10}B can produce states not allowed for the last pair of ^{10}Be and ^{10}C.

It is now useful to recall the study we did with the deuteron in chapter 2, and to observe that the systems p-p, n-n, and the excited state of the deuteron form a triplet of states similar to $T = 1$ above the ground state of the deuteron with $T = 0$. From this point of view it is directly justified that the p-p and n-n systems do not exist, since the excited state of the deuteron is not bound.

A nucleus characterized by Z and N is also characterized by the isospin component $T_z = (Z - N)/2$. The isospin value T of a given state of this nucleus cannot be smaller than $|T_z|$, since T_z is the projection of $\mathbf{T}$. Observing the example of figure 5.11, we see that the ground states have all the smallest possible values of T for each case, that is, $T = |T_z|$. One can verify that the rule works well for all light nuclei.

5.5 Residual Interaction

With the possible values established for the angular momenta of the levels that are split by the residual interaction, the question that immediately arises is the location in energy of those new levels. The hypothesis of the shell model is, as we have seen, to suppose a small effect of the residual interaction and, in that way, to assume H_{res} in (5.36) as a small perturbation. The energy correction to E (eq. 5.41) is given in first order perturbation theory by

$$\Delta E = \int \Psi^* H_{res} \Psi \, d^3 r, \tag{5.75}$$

that is, ΔE is obtained by the expectation value of the residual interaction, calculated with the nonperturbed wavefunctions.

Let us look for example at the case of two particles in excess of a closed core. We can for this case write the residual Hamiltonian (5.37) as

$$H_{res} = \sum_{i=3}^{A} \sum_{ij=3}^{A} V_{i,j}(\mathbf{r}_i, \mathbf{r}_j) - \sum_{i=3}^{A} U(r_i) + \sum_{i=3}^{A} V_{1,i} - U(r_1) + \sum_{i=3}^{A} V_{2,i} - U(r_2) + V_{1,2}. \tag{5.76}$$

The first two parts refer to the closed core and, if we stipulate that it is inert, they can be ignored. For the four following parts we can suppose to a first approximation that the idea of the model is valid, that is, the interaction of particles 1 and 2 with the core is given by an average potential U. Thus, these parts cancel out and we remain with

$$H_{res} = V_{1,2} \tag{5.77}$$

for use in (5.75).

The interaction potential $V_{1,2}$ of valence nucleons would be, in principle, the nucleon-nucleon potential we studied earlier. That potential is not, however, of immediate application when the particles are bound to the nucleus because the Pauli principle modifies in a drastic way the effect of the interaction for bound particles. To find a nucleon-nucleon potential that works well inside the nucleus, two approaches are usual. In the first, one tries to obtain a potential for bound nucleons from the nucleon potential outside the nucleus. This microscopic treatment is very complicated, and the practical results are not very satisfactory. In the second, a phenomenological treatment, a parametrized form is proposed for the interaction and the parameters are determined by a comparison with experimental data. Other conditions of importance for the choice of an effective potential are the simplicity of its use and the elements of the physics of the problem that it incorporates. An example is the *surface delta interaction* (SDI),

$$V(1,2) = A\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(|\mathbf{r}_1| - R_0), \tag{5.78}$$

which tries to incorporate the facts that: 1) the interaction is of short range and 2) inside the nucleus the nucleons are practically free due to the action of the Pauli principle, so the interaction should be located at the nuclear surface of radius R_0. The presence of the delta function makes the calculation of the matrix elements (5.75) relatively simple.

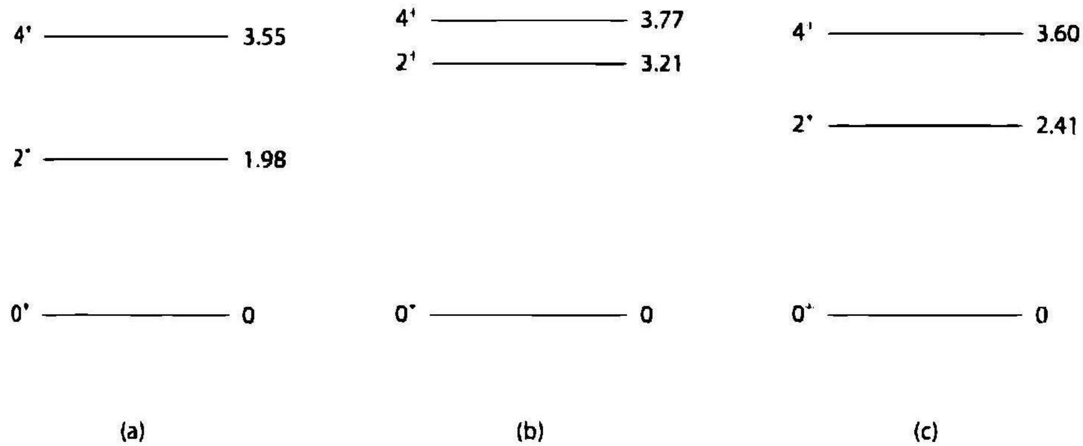

Figure 5.12 Ground state and first excited states of $^{18}_{8}O$: (a) experimental values of the energy (MeV); (b) calculated with the configuration $1d_{\frac{5}{2}}$; (c) calculated with a mixture of the configurations $1d_{\frac{5}{2}}$, $2s_{\frac{1}{2}}$, and $1d_{\frac{3}{2}}$.

Let us take as a practical example $^{18}_{8}O$. For the shell model this nucleus is constituted of a doubly closed core of 8 protons and 8 neutrons plus two valence neutrons in the level $1d_{\frac{5}{2}}$. In agreement with table 5.5, the neutron interaction will split the level $j = \frac{5}{2}$ into three states, of $j = 0, 2$, and 4, all with isotopic spin $T = 1$ (two neutrons) and positive parity ($l = 2$). The corresponding levels appear in the experimental spectrum of $^{18}_{8}O$ (figure 5.12a).

Brussaard and Glaudemans [BG77] calculated the matrix elements for this nucleus using a modified surface delta interaction (MSDI),

$$V(1,2) = A\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(|\mathbf{r}1| - R_0) + B[\mathbf{t}^{(1)} \cdot \mathbf{t}^{(2)}] + C, \tag{5.79}$$

which is obtained from SDI by adding a constant part and another with the Heisenberg operator (3.16), substantially increasing the agreement with experience. The parameters A, B, and C are given by using (5.79) to determine a great number of binding and excitation energies in a certain region of masses, assuming that these parameters are constant in that region. The values of the matrix elements determined for the three levels

$$\begin{aligned}
\left\langle \left(d_{\frac{5}{2}}\right)^2 |V_{12}| \left(d_{\frac{5}{2}}\right)^2 \right\rangle_{J=0,T=1} &= -2.78 \text{ MeV}, \\
\left\langle \left(d_{\frac{5}{2}}\right)^2 |V_{12}| \left(d_{\frac{5}{2}}\right)^2 \right\rangle_{J=2,T=1} &= 0.43 \text{ MeV}, \\
\left\langle \left(d_{\frac{5}{2}}\right)^2 |V_{12}| \left(d_{\frac{5}{2}}\right)^2 \right\rangle_{J=4,T=1} &= 0.99 \text{ MeV},
\end{aligned} \tag{5.80}$$

establish the values of the excitation energy of the states 2^+, $E_{2^+} = 0.43 - (-2.78) = 3.21$ MeV, and 4^+, $E_{4^+} = 0.99 - (-2.78) = 3.77$ MeV, which appear in figure 5.12b. The value of the binding energy of ^{18}O can be obtained through the experimental value of the binding energy of ^{16}O:

$$B(^{18}O) = B(^{16}O) + 2S_n(d_{\frac{5}{2}}) - \left\langle (d_{\frac{5}{2}})^2 |V_{12}| (d_{\frac{5}{2}})^2 \right\rangle_{J=0}, \tag{5.81}$$

where S_n is the separation energy of a neutron of the level $d_{\frac{5}{2}}$ that can, in turn, be determined by (4.10),

$$S_n = B(^{17}O) - B(^{16}O). \tag{5.82}$$

from the experimental values $B(^{17}O) = 131.77$ MeV and $B(^{16}O) = 127.62$ MeV. With these values the result of 5.81 is $B(^{18}O) = 138.7$ MeV, in good agreement with the experimental value $B(^{18}O)_{exp}= 139.8$ MeV.

The above results were obtained supposing the two neutrons permanently in $1d_{\frac{5}{2}}$, that is, the *configuration space* was limited to that configuration. If we compare figures 5.12 (a) and (b) we will see that the excitation energy is poorly determined, not leading to a group of parameters that reproduce the relative distance of the levels 2^+ and 4^+. This is a direct consequence of the limitation of our configuration space and, in fact, there is no reason not to admit that the two neutrons can spend part of their time in other configurations close in energy. When we take into account a group of possible configurations for the valence nucleons, we are making a *configuration mixing* and, with that resource, the results are substantially better. Part (c) of figure 5.12 exhibits the same excited states of ^{18}O but this time using a mixture of the configurations $1d_{\frac{5}{2}}$, $2s_{\frac{1}{2}}$ and $1d_{\frac{3}{2}}$. The agreement with experiment is clearly better.

This approach has a price, however. As we increase the number of configurations, especially if there are more than two valence nucleons, we have to solve the problem of the diagonalization of matrices of dimensions that can reach thousands. This requires sufficiently fast computers, and progress of shell model calculations has, in fact, been conditioned to progress in computer science.

5.6 Nuclear Vibrations

We shall now study collective excitation modes that affect the nucleus as a whole and not just a few nucleons. The model in this section assumes that the nuclear surface can accomplish oscillations around an equilibrium form in the same way as happens with a liquid drop. The starting point is to imagine that a point on the surface of the nucleus is now defined by its radial coordinate $R(\theta, \phi, t)$, a function of the polar and azimuthal angles and of time. Let us consider, for simplicity, oscillations around a spherical form that do not alter the volume and the nuclear density. To describe the form of the nucleus at each instant it will be very convenient to use the property that a function of two variables can be expanded in an infinite series of spherical harmonics and write:

$$R(\theta, \phi, t) = R_0 \left[1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\mu=+\lambda} \alpha_{\lambda\mu}(t) Y_\lambda^\mu(\theta, \phi) \right], \tag{5.83}$$

where the dependence on time is transferred to the coefficients of the expansion.

The application of (5.83) to our problem immediately imposes, conditions on the possible values of λ. Thus, a first examination of this equation shows that the term $\lambda = 0$ just corresponds to a change in the radius of a spherical form, and that is against our

The equations for the mesonic fields are also obtained from the Euler-Lagrange equations, leading to the Klein-Gordon equations with source terms involving the baryon densities:

$$\begin{aligned}
\left[-\Delta + m_\sigma^2\right]\sigma(\mathbf{r}) &= -g_\sigma\,\rho_s(\mathbf{r}) - g_2\sigma^2(\mathbf{r}) - g_3\sigma^3(\mathbf{r}),\\
\left[-\Delta + m_\omega^2\right]\omega_0(\mathbf{r}) &= g_\omega\,\rho_v(\mathbf{r}),\\
\left[-\Delta + m_\rho^2\right]\rho_0(\mathbf{r}) &= g_\rho\,\rho_3(\mathbf{r}),\\
-\Delta\,A_0(\mathbf{r}) &= e\,\rho_c(\mathbf{r}).
\end{aligned} \tag{5.154}$$

The corresponding source terms are

$$\begin{aligned}
&\rho_s = \sum_{i=1}^{A} \overline{\psi}_i\,\psi_i, \qquad \rho_v = \sum_{i=1}^{A} \psi_i^\dagger \psi_i,\\
&\rho_3 = \sum_{p=1}^{Z} \psi_p^\dagger \psi_p - \sum_{n=1}^{N} \psi_n^\dagger \psi_n, \qquad \rho_c = \sum_{p=1}^{Z} \psi_p^\dagger \psi_p.
\end{aligned} \tag{5.155}$$

These sets of equations, known as RMF (relativistic mean field) equations, are solved self-consistently by iteration, as in the usual H-F procedure.

A typical set of parameters are

$$\begin{aligned}
m_\sigma &= 504.89, & g_\sigma &= 9.111,\\
m_\omega &= 780, & g_2 &= -2.304\ \text{fm}^{-1},\\
m_\rho &= 763, & g_3 &= -13.783,\\
& & g_\omega &= 11.493,\\
& & g_\rho &= 5.507,
\end{aligned} \tag{5.156}$$

where the masses are in MeV.

The Lagrangian density (5.144) is based on the hypothesis that the one-pion-exchange potential contribution to the bulk properties of nuclear matter largely averages to zero [SW86].

5.11 Exercises

1. Using a table of masses, determine the percent error in the calculation of the masses of ^{4}He, ^{120}Sn, and ^{208}Pb from (5.8).

2. The nucleus $^{132}_{50}$Sn is not stable, in spite of possessing magic numbers of protons and neutrons. Verify if (5.8) can give an explanation for that fact.

3. Show that the average kinetic energy of the nucleons in the interior of a nucleus given by the Fermi gas model is about 23 MeV.

4. Show that the total kinetic energy of a nucleus with $N = Z$ in the Fermi gas model is given by

$$E_T = 2C_3 A^{-2/3} \left(\frac{A}{2}\right)^{5/3},$$

and find the value of C_3. Repeat the calculation for $N \neq Z$ and show that, in this case,

$$E_T' = C_3 A^{-2/3} (N^{5/3} + Z^{5/3}).$$

5. The group of stable odd-odd nuclei consists of very light nuclei. Is it possible to find a justification for this?

6. The table below shows nuclei with their respective experimental values of spin and parity of the ground state. Compare with the predictions of the extreme shell model for these nuclei and try to justify the discrepancies.

^{7}Be	^{17}F	^{61}Cu	^{91}Zr	^{93}Ni	^{123}Sb	^{159}Tb	^{183}Ta	^{199}Tl	^{209}Pb
$\frac{3}{2}^-$	$\frac{5}{2}^+$	$\frac{3}{2}^-$	$\frac{5}{2}^+$	$\frac{9}{2}^+$	$\frac{7}{2}^+$	$\frac{3}{2}^+$	$\frac{7}{2}^+$	$\frac{1}{2}^+$	$\frac{1}{2}^-$

7. The table below exhibits the orbits attributed to the extra proton and neutron for a series of odd-odd nuclei. a) Try to justify these properties using figure 5.9. b) Determine the spins and parities of these nuclei with help of the Nordheim rule and compare with the experimental values, also shown in the table.

Nucleus	p	n	Spin^P	Nucleus	p	n	Spin^P
^{16}N	$p_{1/2}$	$d_{5/2}$	2^-	^{70}Ga	$p_{3/2}$	$p_{1/2}$	1^+
^{34}Cl	$d_{3/2}$	$d_{3/2}$	0^+	^{90}Y	$p_{1/2}$	$5/2$	2^-
^{38}Cl	$d_{3/2}$	$f_{7/2}$	2^-	^{92}Nb	$g_{9/2}$	$5/2$	7^+
^{41}Sc	$f_{7/2}$	$f_{7/2}$	0^+	^{206}Tl	$s_{1/2}$	$p_{1/2}$	0^-
^{62}Cu	$p_{3/2}$	$f_{5/2}$	1^+	^{202}Bi	$h_{9/2}$	$f_{7/2}$	5^+

8. The spin and parity of ^{9}Be and ^{9}B are $\frac{3}{2}$ for both nuclei. Assuming that these values are given by the last nucleon, justify the observed value, 3^+, of ^{10}B. What other combinations of spin-parity can appear? Verify in a nuclear chart the presence of excited states of ^{10}B that could correspond to those combinations.

9. ^{13}C and ^{13}N both have a ground state $\frac{1}{2}^-$ and three excited states below 4 MeV, of spin-parity $\frac{1}{2}^+$, $\frac{3}{2}^-$, and $\frac{5}{2}^+$. The other states are located above 6 MeV. Interpret these four states using the shell model.

10. The numbers 28 and 40 are sometimes treated as *semi-magic*. Would it be possible, examining figure 5.9, to find a justification for that attribute?