Appendix B Angular Momentum Coupling

B.1 Tensor Operators

B.1.1 Transformation of operators

If the state vectors $|\psi\rangle$ are transformed by a unitary transformation U into $|\psi'\rangle = U|\psi\rangle$, and the operators O are transformed according to

$$O \Rightarrow O' = UOU^{-1}, \tag{B.1}$$

all physical amplitudes are preserved,

$$\langle\psi_2'|O'|\psi_1'\rangle = \langle\psi_2|U^{-1}UOU^{-1}U|\psi_1\rangle = \langle\psi_2|O|\psi_1\rangle. \tag{B.2}$$

It means that the new operators O' play exactly the same role in the new conditions as the old operators O did before the transformation. In other words, when applied to rotation ($U \Rightarrow R$), we made the operators rotate along with the system so that physical measurements by the rotated tools give the same results.

The operators can be classified by their behavior under rotations in the same way that the state vectors were subdivided into rotational multiplets according to their transformation properties. The set of $2J+1$ operators T_{JM}, where J is an integer or half-integer and $M = -J, -J+1, \ldots, J$, is said to form a *tensor operator* of rank J if the operators of the set are transformed under rotations according to the same rules (A.75) as the state vectors $|JM\rangle$,

$$RT_{JM}R^{-1} = \sum_{M'} D^J_{M'M}(R)T_{JM'}. \tag{B.3}$$

For integer $J = l$, the tensor operators T_{lm} have to transform as spherical functions Y_{lm}. In the case of a spinless particle, the tensor operators $T_{lm}(\mathbf{r})$, which are functions of

coordinates, should have the same angular dependence as $Y_{lm}(\mathbf{n})$,

$$T_{lm}(\mathbf{r}) = t_l(r)\,Y_{lm}(\mathbf{n}), \tag{B.4}$$

where the radial factor $t_l(r)$ is the same for all m. In this case it is easy to check directly that the transformation rule (B.3) preserves amplitudes (B.4). Indeed, as we know from (A.16) and (A.101), the transformation (B.3) for the function of coordinates (B.4) should give $T_{lm}(R^{-1}\mathbf{n})$. (Remember that here we transform an operator; the first factor R on the left-hand side of (B.3) transforms only T_{JM} and cancels R^{-1} so that all functions placed after T_{JM} are not affected.) The transformed matrix element (B.2) differs from the original one merely by the change of angular variables $\mathbf{n} \to R^{-1}\mathbf{n}$ in the integrand, which cannot change the integral. A similar conclusion holds for the functions of the momentum, which will be proportional to $Y_{lm}(\mathbf{n_p})$, where $\mathbf{n_p} = \mathbf{p}/|\mathbf{p}|$.

B.1.2 Scalars and vectors

Any function of coordinates can be represented by a series of spherical harmonics with coefficients depending only on r. This is equivalent to an expansion over irreducible tensor operators. The lowest term, $l = 0$ or *s-wave*, is proportional to the spherical function

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \tag{B.5}$$

and does not depend on angles. It is not affected by rotations ($T_{00} \sim Y_{00}$ is *scalar*).

There are 3 *p-wave* functions, $l = 1$,

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}. \tag{B.6}$$

For any vector $\mathbf{V}$ we can introduce, instead of Cartesian components $V_i = (V_x, V_y, V_z)$, the so-called *spherical* components V_m, $m = 0, \pm1$:

$$V_0 = V_z, \quad V_{\pm1} = \mp\frac{1}{\sqrt{2}}(V_x \pm iV_y). \tag{B.7}$$

Note that the spherical components $J_{\pm1}$ of the angular momentum differ only by a factor $\mp1/\sqrt{2}$ from the lowering and raising operators $J_\pm$, (A.61). From (B.6) and (B.7) we see that the functions $Y_{1m}(\mathbf{n})$ are essentially the spherical components of the vector $\mathbf{n}$,

$$Y_{1m}(\mathbf{n}) = \sqrt{\frac{3}{4\pi}}\,n_m. \tag{B.8}$$

The scalar product of the vectors in Cartesian coordinates can be also expressed in terms of the spherical components (B.7):

$$(\mathbf{a}\cdot\mathbf{b}) = \sum_{m=0,\pm1} (-)^m a_m b_{-m}. \tag{B.9}$$

Note, that for unit vectors (B.9) is merely a particular case of the addition theorem (A.112) for $l = 1$.

All vectors behave in the same way under rotations. Therefore we conclude that any vector operator is a tensor of rank 1. The coordinate vector $\mathbf{n}$ is an example of a *polar* vector. Its components, together with the spherical functions Y_{1m}, change sign under spatial inversion.

B.1.3 Tensors of rank 2

Let us consider 9 quantities $T_{ij} = a_i b_j$ constructed of the Cartesian components of vectors $\mathbf{a}$ and $\mathbf{b}$. They are *reducible* under rotations and can be grouped in smaller *irreducible* sets. First we separate two parts with different symmetry (submatrices which are symmetric, S, or antisymmetric, A, under a transposition of matrix indices),

$$T_{ij} = S_{ij} + A_{ij} = \tfrac{1}{2}(a_i b_j + a_j b_i) + \tfrac{1}{2}(a_i b_j - a_j b_i). \tag{B.10}$$

The symmetric part S_{ij} is further reducible since the trace $\mathrm{Tr}\, S = S_{ii} = (\mathbf{a} \cdot \mathbf{b})$ is a scalar. We can subtract the invariant scalar with a coefficient such that the rest are traceless,

$$S_{ij} = \tfrac{1}{3}(\mathbf{a} \cdot \mathbf{b})\delta_{ij} + Q_{ij}. \tag{B.11}$$

The symmetric tensor

$$Q_{ij} = \tfrac{1}{2}\left(a_i b_j + a_j b_i - \tfrac{2}{3}(\mathbf{a} \cdot \mathbf{b})\delta_{ij}\right) \tag{B.12}$$

is traceless, $\mathrm{Tr}\, Q = Q_{ii} = 0$, and irreducible. It has 5 independent components. The antisymmetric part

$$A_{ij} = \tfrac{1}{2}(a_i b_j - a_j b_i) = \tfrac{1}{2}\epsilon_{ijk}[\mathbf{a} \times \mathbf{b}]_k \tag{B.13}$$

has 3 independent components. It is equivalent (with respect to rotations) to a vector $[\mathbf{a} \times \mathbf{b}]$. If both $\mathbf{a}$ and $\mathbf{b}$ are polar vectors (previous paragraph), the components of their vector product do not change sign under inversion of spatial coordinates. Such a vector is called an *axial* or pseudovector; as an example we can recall the orbital momentum (A.3).

Summarizing, the decomposition of the reducible tensor T_{ij} into irreducible parts S_{ij}, A_{ij}, and Q_{ij} can be symbolically presented as

$$\underline{3} \times \underline{3} = \underline{1} + \underline{3} + \underline{5}, \tag{B.14}$$

where the underlined numbers designate the dimensions of representations.

In order to understand the rotational properties of the symmetric tensor (B.12), we have to compare its transformational features with those of spherical harmonics. Since all vectors transform in the same way, it is sufficient to consider the case $\mathbf{a} = \mathbf{b} = \mathbf{n}(\theta, \varphi)$. Then we can establish a one-to-one correspondence between the spherical functions of the second rank $Y_{2m}(\mathbf{n})$ and certain linear combinations of components Q_{ij},

$$Y_{20}(\mathbf{n}) = \sqrt{\frac{5}{16\pi}}(2\cos^2\theta - \sin^2\theta) \Rightarrow \sqrt{\frac{5}{4\pi}}\frac{1}{2}(2Q_{zz} - Q_{xx} - Q_{yy}), \tag{B.15}$$

$$Y_{2\pm1}(\mathbf{n}) = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\varphi} \Rightarrow \sqrt{\frac{5}{4\pi}}\sqrt{\frac{3}{2}}(Q_{xz} \pm iQ_{yz}), \tag{B.16}$$

$$Y_{2\pm2}(\mathbf{n}) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi} \Rightarrow \sqrt{\frac{5}{4\pi}}\sqrt{\frac{3}{8}}(Q_{xx} \pm 2iQ_{xy} - Q_{yy}). \tag{B.17}$$

Inversely, the components Q_{ij} are linear combinations of $Y_{2m}(\mathbf{n})$ and therefore correspond to a tensor operator of the second rank. The five combinations on the right-hand side of (B.15–B.17) are organized in such a way that they form a spherical tensor

$$Q_{2m} \propto \sqrt{\frac{4\pi}{5}}Y_{2m}(\mathbf{n}). \tag{B.18}$$

The conclusion is that any set of 9 quantities T_{ij} which transform under rotations as a product of two vectors can be decomposed into scalar, vector, and symmetric second rank tensor parts. The procedure can be extended for any tensor $T_{ijk\ldots} \sim a_i b_j c_k \ldots$.

B.1.4 Introduction to selection rules

Tensor properties of the operators are important in calculations of physical amplitudes proportional to the matrix elements $\langle\psi'_{J_2M_2}|T_{JM}|\psi_{J_1M_1}\rangle$. For given initial and final multiplets of states, we have here $(2J_2+1)(2J+1)(2J_1+1)$ different matrix elements. However, as we will see, only one number characterizes the relevant physics. The rest are completely determined by geometrical considerations. Some matrix elements vanish exclusively due to the rotational symmetry of the states and the operators, others turn out to be strongly interrelated.

The simplest selection rules can be discovered directly from the definition of tensor operators (B.3). Let us consider an infinitesimal rotation through an angle $\delta\alpha$ around the axis $\mathbf{n}$. The corresponding operator is $R = 1 - i(\mathbf{J}\cdot\mathbf{n})\delta\alpha$, (A.11). Keeping linear terms in $\delta\alpha$, the left-hand side of (B.3) is expressed via the commutator of the tensor T_{JM} with the angular momentum

$$RT_{JM}R^{-1} = T_{JM} - i\delta\alpha[(\mathbf{J}\cdot\mathbf{n}), T_{JM}]. \tag{B.19}$$

In a given representation the matrix elements (A.70) of this rotation are

$$D^J_{M'M}(R) = \delta_{M'M} - i\delta\alpha\langle JM'|(\mathbf{J}\cdot\mathbf{n})|JM\rangle. \tag{B.20}$$

Since the axis direction $\mathbf{n}$ is arbitrary, (B.19) implies the commutation relation valid for any tensor operator,

$$[J_m, T_{JM}] = \sum_{M'}\langle JM'|J_m|JM\rangle T_{JM'} \tag{B.21}$$

where J_m can be either Cartesian or spherical, (B.7).

The $J_z = J_0$ component of (B.21) reads

$$[J_z, T_{JM}] = MT_{JM}. \tag{B.22}$$

This is a typical ladder relation, recall (A.41). We conclude that, acting on a state with a certain z-projection of the total angular momentum of a system, a tensor operator T_{JM} *raises* this projection by M. We obtained the simple *selection rule*: in the transitions $\langle a_2 J_2 M_2 | T_{JM} | a_1 J_1 M_1 \rangle$, where a_1 and a_2 are symbols for all additional (nonrotational) quantum numbers, the only nonzero amplitudes are those with $\Delta J_z \equiv M_2 - M_1 = M$,

$$T_{JM}: \quad \Delta J_z = M. \tag{B.23}$$

Our notation in (B.7) for the spherical components of vectors agrees with this general rule. The result does not depend on the specific values of J_1, J_2, or other (nonrotational) quantum numbers a_1, a_2.

The raising, $m = +1$, component of (B.21) contains on the right-hand side the only term $M' = M + 1$. We see that the operator product of J_{+1}, which has a selection rule $\Delta J_z = +1$, and T_{JM} creates a new operator corresponding to $\Delta J_z = M + 1$. For the lowering component of (B.21), $m = -1$ and $\Delta J_z = M - 1$. In the product of operators, the selection rules for the projection J_z are simply added *algebraically*.

When applied to a vector operator, $T_{1M} \to V_M$, the general equation (B.22) gives $[J_z, V_{\pm 1}] = \pm V_{\pm 1}$. Obviously, the rotation around the z-axis does not change the z-component of a vector, $[J_z, V_z] = 0$. In Cartesian coordinates, such relations are equivalent to the commutation relation

$$[J_k, V_l] = i\epsilon_{kln} V_n, \tag{B.24}$$

which generalizes the angular momentum algebra (A.59) for an arbitrary vector. All such commutation rules are of pure geometrical origin and therefore universally valid regardless of specific nature, or behavior under inversion, of the tensor operator.

B.2 Angular Momentum Coupling

B.2.1 Two subsystems

Consider two subsystems with angular momenta j_1 and j_2. The total quantum space contains $d = (2j_1 + 1)(2j_2 + 1)$ states obtained by combinations of various members of multiplets $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$ with projections $m_1 = -j_1, \ldots, j_1$ and $m_2 = -j_2, \ldots, j_2$, respectively. Those basis states can be designated as

$$|j_1 m_1; j_2 m_2\rangle. \tag{B.25}$$

If the subsystems do not interact, all four quantum numbers j_1, j_2, m_1, m_2 are conserved (we assume the rotational invariance of the whole system). Then it is convenient to use the basis states of independent subsystems. Each system can be rotated separately according to its angular momentum operators $\mathbf{j}_1$ and $\mathbf{j}_2$, generating corresponding transformations. One can imagine the picture of separate precession of the constituent angular momenta around the common axis.

We can characterize the system in a different way by probing its behavior under common rotation when the subsystems are rotated together. The generator of such rotations is the total angular momentum

$$\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2. \tag{B.26}$$

In the previous picture of separate precessions the operator $\mathbf{J}$ has no certain value because the result of the vector addition (B.26) depends on the instantaneous mutual orientation of $\mathbf{j}_1$ and $\mathbf{j}_2$. The states (B.25) are superpositions of states with definite values of $\mathbf{J}^2$. In the case of interacting subsystems the separate rotations in general violate structure, which makes states (B.25) nonstationary, whereas common rotations preserve the intrinsic structure. Then it is more convenient to describe the states by the quantum numbers J and M related to the generator (B.26) of total rotations, even if both descriptions use the complete set of states being therefore mathematically equivalent.

With respect to common rotations, when the relative orientation of the subsystems is kept intact and they rotate as a whole, the complete set of the states (B.25) is *reducible*. Any possible relative orientation will give rise to a multiplet $|JM\rangle$ of states transforming between each other under common rotations. We first define the relative orientation and corresponding total momentum $\mathbf{J}$ (the angular momenta of subsystems precess around $\mathbf{J}$) and then allow the total construction to rotate around the space-fixed quantization axis that defines the projection M. The z-projections m_1 and m_2 cease to conserve (but absolute values j_1 and j_2 still do because we do not change the internal structure of the subsystems). so that we obtain the new set of states

$$|j_1 j_2; JM\rangle \tag{B.27}$$

that form multiplets *irreducible* under common rotations. For individual angular momenta of the subsystems the effective quantization axis is now that of the total vector $\mathbf{J}$. Indeed. as seen from eq. (B.26), the state (B.27) has certain projections

$$(\mathbf{j}_1 \cdot \mathbf{J}) = \frac{J(J+1) + j_1(j_1+1) - j_2(j_2+1)}{2}, \tag{B.28}$$

and similarly for $(\mathbf{j}_2 \cdot \mathbf{J})$.

The relative orientations allowed in quantum mechanics are quantized in space. Therefore the possible total momentum J, (B.26), can take only a finite discrete set of (positive) values. In any case, the new states (B.27), where each multiplet contains $2J+1$ members. should be as complete as the old set (B.25), so their dimensions have to coincide:

$$d = \sum_J (2J+1) = d_1 d_2 = (2j_1+1)(2j_2+1). \tag{B.29}$$

B.2.2 Decomposition of reducible representations

First we have to find all irreducible representations that taken together span the whole space (B.25). It can be done by a simple construction which is equivalent to the standard

group theory procedure of finding the characters of the representations (traces of the matrix D^J).

Put all basis states (B.25) into a $d_1 \times d_2$ table that has d_1 vertical columns numbered by m_1, $-j_1 \le m_1 \le j_1$, and d_2 horizontal rows numbered by m_2, $-j_2 \le m_2 \le j_2$. For definiteness, assume $j_1 \ge j_2$. Each state (square of the table) has a certain value

$$M = m_1 + m_2 \tag{B.30}$$

of the total projection $J_z = j_{1z} + j_{2z}$, (B.26). Any state $|JM\rangle$ of the set (B.27) will be a superposition of states lying on a straight diagonal line (B.30) corresponding to a given M-value. The number of squares on this line is equal to the number of multiplets (B.27) that include this value of the projection, that is, with angular momentum $J \ge M$.

Start with the upper right corner, $M = j_1 + j_2$. This is the maximum possible total projection. It is constructed uniquely (alignment of constituent momenta). There is only one multiplet where this value of M gives the maximum projection so this state has the highest possible magnitude of $J_{\max} = M_{\max} = j_1 + j_2$. This highest multiplet should have all other members, $M = J - 1, J - 2, \ldots, -J = -(j_1 + j_2)$ as well.

Let us come to the next diagonal line $M = J_{\max} - 1$. There are two such states. They can form two linearly independent combinations. One of them belongs to the highest multiplet as was mentioned in the paragraph above. This combination $|J_{\max}\ M = J_{\max} - 1\rangle$ can be obtained by the action of the lowering operator

$$J_- = j_{1-} + j_{2-} \tag{B.31}$$

on the maximum aligned state (recall that the components of $\mathbf{J}$ act only within the multiplet). According to (A.71), the result is

$$|J = j_1 + j_2\ M = j_1 + j_2 - 1\rangle = \sqrt{2j_1}|j_1 j_1 - 1; j_2 j_2\rangle + \sqrt{2j_2}|j_1 j_1; j_2 j_2 - 1\rangle. \tag{B.32}$$

On the other hand, this should be equal to the action of the total J_-,

$$J_-|J = j_1 + j_2\ M = j_1 + j_2\rangle = \sqrt{2(j_1 + j_2)}|J = j_1 + j_2\ M = j_1 + j_2 - 1\rangle. \tag{B.33}$$

The comparison of the last two expressions defines

$$|J = j_1 + j_2\ M = j_1 + j_2 - 1\rangle = \sqrt{\frac{j_1}{j_1 + j_2}}\left|j_1 j_1 - 1; j_2 j_2\right\rangle + \sqrt{\frac{j_2}{j_1 + j_2}}\left|j_1 j_1; j_2 j_2 - 1\right\rangle. \tag{B.34}$$

The second possible combination along the same short diagonal with $M = j_1 + j_2 - 1$ is the highest for the second multiplet. Therefore we open the new multiplet and see that the value of the total momentum $J = j_1 + j_2 - 1$ is also possible. This state has another J and has to be orthogonal to the state (B.33), although they have the same value of M. By orthogonality, we find

$$|J = j_1 + j_2 - 1\ M = j_1 + j_2 - 1\rangle = \sqrt{\frac{j_2}{j_1 + j_2}}\left|j_1 j_1 - 1; j_2 j_2\right\rangle - \sqrt{\frac{j_1}{j_1 + j_2}}\left|j_1 j_1; j_2 j_2 - 1\right\rangle. \tag{B.35}$$

Here we can add an arbitrary extra phase, for example change the common sign—this is a matter of convention.

The next step on the way down shows three states with $M = j_1 + j_2 - 2$. Two of the three combinations belong to the previous multiplets while the third one opens the new multiplet with $J = j_1 + j_2 - 2$. This procedure is obvious and regular. Each step to a lower diagonal line adds a new multiplet with angular momenta steadily decreasing. This takes place for the last time when we reach the main diagonal, which corresponds to $M = j_1 - j_2$. At this step we open the multiplet with the lowest possible angular momentum $J_{\min} = j_1 - j_2$. After that the number of possible M does not increase which means that we just fill up the available multiplets. The lowest multiplet will be full at the line reaching the left upper corner of the table. Later on, each next step one completes one of the multiplets until we arrive at the left lower corner with only one state $M = -M_{\max} = -j_1 - j_2$ which completes the largest multiplet $J = j_1 + j_2$.

We can summarize the result of this exercise saying that possible values of the total angular momentum J in the vector coupling of subsystems with angular momenta j_1 and j_2 are

$$|j_1 - j_2| \le J \le j_1 + j_2. \tag{B.36}$$

Each value of J appears only once and it is easy to check the fulfillment of (B.29): we used all squares of our table in the rearrangement of the reducible space (B.25) into irreducible multiplets (B.27). While projections (B.30) are added algebraically, the magnitudes of angular momenta are added geometrically; inequalities (B.36) give exactly the same boundaries that would be valid for the addition of two Euclidean vectors (*triangle conditions*). However, quantum mechanics put an extra constraint of space quantization for the total angular momentum whose allowed values (B.36), in accordance with general rules for the SU(2) group, are all integer or half-integer depending on the values of j_1 and j_2.

B.2.3 Tensor operators and selection rules revisited

Electric and magnetic multipoles are typical examples of operators forming sets of $2\lambda + 1$ quantities $T_{\lambda\mu}$ that are closed with respect to the rotation group. Under rotations such quantities are transformed into linear combinations of quantities belonging to the same set, and the rule of transformation is exactly the same as for the spherical functions $Y_{\lambda\mu}$. Such a set of operators are said to form a *tensor operator* of rank λ. The physical consequences that follow from geometrical considerations are analogous for all tensor operators of the same rank, regardless of their physical nature.

In the case of the operator proportional to a spherical function $Y_{\lambda\mu}$, its action on a state $|J_1 M_1\rangle$ can be considered as a vector coupling of angular momenta of two "subsystems," $\mathbf{J}_1$ of the state and $\vec{\lambda}$ of the operator. According to the rules of the rotation group, the final angular momentum

$$\mathbf{J}_2 = \mathbf{J}_1 + \vec{\lambda} \tag{B.37}$$

can take all values J_2 which differ by one unit within the limits put by the triangle condition (B.36),

$$|J_1 - \lambda| \le J_2 \le J_1 + \lambda. \tag{B.38}$$

The projections of angular momenta are added algebraically, (B.30),

$$M_2 = M_1 + \mu. \tag{B.39}$$

In fact, the triangle conditions (B.38) are symmetric with respect to all three angular momenta J_1, J_2 and λ.

Equations (B.38)–(B.39) determine selection rules which are exactly the same for any tensor operator $T_{\lambda\mu}$: matrix elements $\langle a_2 J_2 M_2 | T_{\lambda\mu} | a_1 J_1 M_1 \rangle$ of a tensor operator between any states with certain angular momentum quantum numbers (and arbitrary additional quantum numbers a_1, a_2) can be different from zero if and only if the conditions (B.38) and (B.39) are fulfilled. For example, multipole transitions of multipolarity λ are strictly forbidden if $\Delta J = |J_2 - J_1| > \lambda$ or $\lambda > J_1 + J_2$.

As a particular case, the angular momentum selection rules restrict multipole moments that are allowed to have nonvanishing *expectation values* in a state with angular momentum J. Here we are interested in the *diagonal* elements $J_1 = J_2 = J$. The rule (B.38) shows that the allowed multipoles are those of rank λ satisfying

$$0 \le \lambda \le 2J. \tag{B.40}$$

As follows from (B.40), a system with angular momentum $J = 0$ accepts $\lambda = 0$ only, and therefore can have nonzero charge (2.9) but none of higher multipoles. A system with spin $\frac{1}{2}$, such as the nucleon or electron, can have $\lambda = 0$ or 1, that is, charge and dipole moments, electric (2.11) or magnetic (2.13). A nonzero quadrupole moment, $\lambda = 2$, appears only for systems with $J \ge 1$.

B.2.4 Vector coupling of angular momenta

We have found that in quantum mechanics two subsystems with rotational quantum numbers j_1, m_1 and j_2, m_2 being coupled together can form systems with various quantum numbers j_3, m_3 with respect to their rotation *as a whole*. The probability amplitudes of different possible outputs j_3, m_3 of the vector coupling are given by the *Clebsch-Gordan coefficients* (CGC) $\langle j_3 m_3 | j_1 m_1; j_2 m_2 \rangle$:

$$|j_1 m_1; j_2 m_2\rangle = \sum_{j_3 m_3} \langle j_3 m_3 | j_1 m_1; j_2 m_2 \rangle |(j_1 j_2) j_3 m_3\rangle, \tag{B.41}$$

where the last notation reminds us of the angular momenta (j_1, j_2) of the constituents. The allowed values of j_3, m_3 in (B.41) are given by the same selection rules (B.38), (B.39).

The CGC perform a transformation between two possible sets of basis states (two separate subsystems and the combined system). Both sets are complete, orthonormalized, and equally good, albeit either could be more or less convenient in a given physical situation. The transformation from one set to another one is *unitary*, so that the coefficients $\langle j_1 m_1; j_2 m_2, | j_3 m_3 \rangle$, which perform the inverse transformation,

$$|(j_1 j_2) j_3 m_3\rangle = \sum_{m_1 m_2} \langle j_1 m_1; j_2 m_2 | j_3 m_3 \rangle | j_1 m_1; j_2 m_2 \rangle, \tag{B.42}$$

are complex conjugate with respect to those in (B.41). At the standard choice of phases

for the matrix elements of the angular momentum, the CGC are *real*, and we will use the notation

$$C^{j_3 m_3}_{j_1 m_1 j_2 m_2} = \langle j_1 m_1; j_2 m_2 | j_3 m_3 \rangle = \langle j_3 m_3 | j_1 m_1; j_2 m_2 \rangle. \tag{B.43}$$

The orthonormalization conditions for both sets of states give

$$\sum_{m_1 m_2} C^{j_3 m_3}_{j_1 m_1 j_2 m_2} C^{j_3' m_3'}_{j_1 m_1 j_2 m_2} = \delta_{j_3 j_3'} \delta_{m_3 m_3'} \tag{B.44}$$

and

$$\sum_{j_3 m_3} C^{j_3 m_3}_{j_1 m_1 j_2 m_2} C^{j_3 m_3}_{j_1 m_1' j_2 m_2'} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}. \tag{B.45}$$

B.2.5 Wigner-Eckart theorem

We have worked out the selection rules for the tensor operator $T_{\lambda\mu}$ related to rotational invariance. For the angular momenta J_2, J_1, and λ satisfying (B.38) and (B.39), in general there are many nonzero matrix elements (we indicate explicitly other quantum numbers of the states which are fixed for a given set of matrix elements)

$$\langle a_2 J_2 M_2 | T_{\lambda\mu} | a_1 J_1 M_1 \rangle. \tag{B.46}$$

All matrix elements with different combinations of projections contain the same physics, differing in the mutual orientation of the states $|a_1 J_1 M_1\rangle$, $|a_2 J_2 M_2\rangle$ and of the probe $T_{\lambda\mu}$. That is why, for instance, in tables of physical quantities one can find only one number for a magnetic moment of a particle or of a nucleus instead of the set of numbers corresponding to various matrix elements $\langle JM' | \mu_\mu | JM \rangle$. It is possible to separate the universal geometric information from the specific characteristics of a system under study.

Let us consider the action of the tensor operator $T_{\lambda\mu}$ onto the initial state $|a_1 J_1 M_1\rangle$. As a result of the vector coupling of the angular momenta (B.37), one can obtain for the intermediate state only the angular momentum projection $M' = M + \mu$ and the magnitude of angular momentum J' allowed by the triangle conditions $\mathbf{J}' = \mathbf{J} + \vec{\mu}$. The relative amplitudes of possible intermediate states $|J'M'\rangle$ are given by the CGC as in (B.41),

$$T_{\lambda\mu} |a_1 J_1 M_1\rangle = \sum_{J'M'} C^{J'M'}_{\lambda\mu J_1 M_1} |a_1 (T_\lambda J_1) J' M'\rangle, \tag{B.47}$$

Now we have to project the state $|J'M'\rangle$ onto the final state $|a_2 J_2 M_2\rangle$. Because of the orthogonality of eigenfunctions corresponding to different eigenvalues of Hermitian operators, only the term $J' = J_2, M' = M_2$ in the sum (B.47) survives. Moreover, the matrix element (B.46) cannot change if the initial state, final state, and operator are undergoing a *common* rotation. Therefore the result of the last projection $\langle a_2 J_2 M_2 | J'M' \rangle$ does not depend on the specific value of $M' = M_2$.

We came to the important conclusion: in any matrix element (B.46) of a tensor operator between the states with a certain angular momentum and its projection, the entire dependence on the magnetic quantum numbers M_1, μ and M_2 enters through the CGC only. The

remaining factor does not carry any M-dependence and characterizes the physical amplitude of the process, regardless of the orientation of the system. Note that all rotational selection rules are already included in this CGC. This is the essence of the *Wigner-Eckart theorem.*

Using the $3j$ symbol instead of the CGC, we write the result as

$$\langle a_2 J_2 M_2 | T_{\lambda\mu} | a_1 J_1 M_1 \rangle = (-)^{J_2 - M_2} \begin{pmatrix} J_2 & \lambda & J_1 \\ -M_2 & \mu & M_1 \end{pmatrix} \langle a_2 J_2 \| T_\lambda \| a_1 J_1 \rangle. \tag{B.48}$$

Here the M-independent factor is introduced as a double-barred (*reduced*) matrix element. The phase factor for the final state in (B.48) is in accordance with the arguments related to time conjugation: the final state ($M_2 = \mu + M_1$) has to be reversed to make the situation symmetric. We see that the geometric part of information is factored into the $3j$-symbol while the intrinsic orientation-independent physics is concentrated in the reduced matrix element. As we declared above, only one number is sufficient to describe the whole set of matrix elements (B.46) if the rotational quantum numbers of the states and the operators are known.

The number shown in physical tables for the expectation values of multipole operators in a state with angular momentum J is, by convention, taken for the substate with the maximum projection $M = J$. Then $\mu = 0$, and the tabular value is

$$T_\lambda(a, J) \equiv \langle aJJ | T_{\lambda 0} | aJJ \rangle. \tag{B.49}$$

For example, the vector ($\lambda = 1$) component needed in (B.49) is $V_0 = V_z$; recall (B.23). The tabular magnetic moment therefore is the expectation value of its projection onto the quantization axis z in the state with the maximum alignment along the z-axis.

B.2.6 Vector model

The Wigner-Eckart theorem provides us with the justification of a simple procedure used from the very early days of atomic physics for calculating the expectation values, as for example,

$$\langle aJM' | \mathbf{V} | aJM \rangle \tag{B.50}$$

where the initial and final states belong to the same multiplet but may differ by the projection of the angular momentum, and $\mathbf{V}$ is an arbitrary vector operator.

The naive although correct way of reasoning is following. The semiclassical image of the state $|JM\rangle$ is that of precession. The angular momentum vector $\mathbf{J}$ of the length $\sqrt{J(J+1)}$ has a projection M onto the quantization axis and it is precessing around this axis forming a cone with the fixed polar angle θ, $\cos\theta = M/\sqrt{J(J+1)}$. The transverse components $J_{x,y}$ are averaged out and have zero expectation values $\langle J_x \rangle$ and $\langle J_y \rangle$ but nonzero mean square values $\langle J_x^2 \rangle$ and $\langle J_y^2 \rangle$. The sum $\langle J_x^2 + J_y^2 \rangle$ supplements M^2 to the total magnitude $J(J+1)$ of the angular momentum squared. In this situation any vector $\mathbf{V}$ related to the system can be in average aligned along the only available preferential direction, namely that of the

angular momentum **J**. This proportionality of the two vectors can be written as the *vector model*

$$\mathbf{V} = v(a, J)\mathbf{J}, \tag{B.51}$$

where the coefficient of proportionality is a scalar $v(a, J)$ which can depend on the length of the angular momentum and on other characteristics of the state (a), and the equality has to be understood as the equivalence of the two operators for any matrix element *within* the multiplet. We find this factor by taking the projection on **J** in both parts of (B.51):

$$v(a, J) = \frac{\langle(\mathbf{V} \cdot \mathbf{J})\rangle}{\mathbf{J}^2} = \frac{\langle(\mathbf{V} \cdot \mathbf{J})\rangle}{J(J+1)}. \tag{B.52}$$

Instead of this loose derivation, we can use the Wigner-Eckart theorem (B.48). Recalling that any vector is a tensor operator of rank 1 and introducing its spherical components V_μ according to (B.7), we can write the matrix element (B.50) between the states of the same multiplet as

$$\langle aJM'|V_\mu|aJM\rangle = (-)^{J-M'} \begin{pmatrix} J & 1 & J \\ -M' & \mu & M \end{pmatrix} \langle aJ\|V\|aJ\rangle. \tag{B.53}$$

Exactly in the same way, we find for the angular momentum **J**

$$\langle aJM'|J_\mu|aJM\rangle = (-)^{J-M'} \begin{pmatrix} J & 1 & J \\ -M' & \mu & M \end{pmatrix} \langle aJ\|J\|aJ\rangle. \tag{B.54}$$

Eliminating the 3j-symbol, we find that matrix elements of any vector **V** and of the angular momentum **J** are proportional as in the vector model (B.51) with the coefficient

$$v(a, J) = \frac{\langle aJ\|V\|aJ\rangle}{\langle aJ\|J\|aJ\rangle}. \tag{B.55}$$

It is worthwhile to stress again that the whole procedure makes sense only for the transitions *within* the multiplet $|aJM\rangle$. While **J** acts only inside the multiplet, an arbitrary vector **V** can have also off-diagonal elements in J and a matrix elements (B.50) that are unrelated to the matrix elements of **J**.

To establish the final correspondence of (B.52) and (B.55), we calculate the expectation values of scalar quantities $\mathbf{J}^2$ and $(\mathbf{J} \cdot \mathbf{V})$. The calculation is straightforward: write down the scalar product in spherical components (B.9); express the matrix element sought as a product of matrix elements of individual vectors with the summation over the intermediate projection (since at least one of the vectors is **J**, all intermediate states have the same quantum numbers aJ); apply the Wigner-Eckart theorem (B.48) to each of the factors; and sum over intermediate projections. The result is

$$\langle aJM''|(\mathbf{J} \cdot \mathbf{V})|aJM\rangle = \frac{\delta_{M''M}}{2J+1} \langle aJ\|J\|aJ\rangle \langle aJ\|V\|aJ\rangle. \tag{B.56}$$

Here **V** is an arbitrary vector operator. As it should be, the matrix elements of a scalar quantity do not depend on the orientation ($M = M_2$). In the particular case $\mathbf{V} \Rightarrow \mathbf{J}$, the

left-hand side of (B.56) is equal to $\delta_{M''M}J(J+1)$. It defines the reduced matrix element for the angular momentum,

$$\langle aJ\|J\|aJ\rangle^2 = J(J+1)(2J+1). \tag{B.57}$$

Finally, combining these results, we obtain

$$\langle aJM'|V_\mu|aJM\rangle = \frac{\langle aJ|(\mathbf{J}\cdot\mathbf{V})|aJ\rangle}{J(J+1)}\langle aJM'|J_\mu|aJM\rangle, \tag{B.58}$$

which is nothing but the vector model (B.51), (B.52). Since the matrix element of $(\mathbf{J}\cdot\mathbf{V})$ in (B.58) does not depend on M, we do not need to indicate the projections explicitly.