Appendix C Symmetries

C.1 Time Reversal

The equations of quantum as well as classical mechanics are time-reversible if there is no variable in time or magnetic external fields. In contrast to other discrete symmetries, this symmetry does not correspond to any conserved Hermitian operator. Under time reversal, not only do operators of linear and angular momenta have to change sign but the direction of processes has to be reversed as well. A *final* state of a particle with momentum $\mathbf{p}$ and spin $\mathbf{s}$ is to be transformed into the *initial* state of the time-reversed process with momentum $-\mathbf{p}$ and spin $-\mathbf{s}$. Therefore the time reversal operation $\mathcal{T}$ includes transposition (or complex conjugation $\mathcal{K}$) of observables.

Let us define the time reversal operation as

$$\mathcal{T} = U_T \mathcal{K} O_T, \tag{C.1}$$

where O_T changes the time direction in the quantities which are explicitly time-dependent, and U_T is a unitary operator which has to ensure correct transformations of physical quantities. Due to the complex conjugation $\mathcal{K}$, the operator (C.1) is not linear in usual sense; it acts on the coefficients of the linear superposition

$$\mathcal{T}(a\Psi_1 + b\Psi_2) = a^*\mathcal{T}\Psi_1 + b^*\mathcal{T}\Psi_2 \tag{C.2}$$

(sometimes it is called *antilinear*).

When applied to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(t) = H\Psi(t), \tag{C.3}$$

the operation (C.1) gives

$$-i\hbar\frac{\partial}{\partial(-t)}\left(U_T\Psi^*(-t)\right) = \mathcal{T}H\mathcal{T}^{-1}\left(U_T\Psi^*(-t)\right). \tag{C.4}$$

Let us designate the time reversed quantities with a tilde,

$$A \Rightarrow \tilde{A} \equiv \mathcal{T} A \mathcal{T}^{-1}. \tag{C.5}$$

The reversed dynamics (C.4) are governed by the reversed Hamiltonian

$$\tilde{H} = \mathcal{T} H \mathcal{T}^{-1} = U_T H^*(-t) U_T^{-1}. \tag{C.6}$$

As seen from (C.4), the new state vector

$$\tilde{\Psi}(t) = U_T \Psi^*(-t) \tag{C.7}$$

satisfies the same Schrödinger equation (C.3) as the original vector $\Psi(t)$ if the Hamiltonian is $\mathcal{T}$*-invariant,*

$$\tilde{H} = H. \tag{C.8}$$

If the Hamiltonian is time-independent, it has stationary eigenfunctions,

$$\Psi(t) = \Psi(0) e^{-(i/\hbar)Et}. \tag{C.9}$$

If the Hamiltonian is also $\mathcal{T}$-invariant, (C.9) shows that the time-reversed ("time conjugate") function with the amplitude $U_T \Psi^*(0)$ is also an eigenfunction with the same energy E. For a nondegenerate eigenvalue E, there is only one eigenfunction corresponding to this energy so that the time reversed function can differ from the original one only by a constant phase. However, in the degenerate case, one can have two mutually time-reversed linearly independent states Ψ and $\tilde{\Psi}$ with the same energy.

The specific form of U_T depends on the representation used for the description of a specific system. For spinless particles described in the coordinate representation by their coordinates $\mathbf{r}$ and momenta $\mathbf{p} = -i\hbar\nabla$ only, time reversal should give $\mathbf{r} \Rightarrow \mathbf{r}$, $\mathbf{p} \Rightarrow -\mathbf{p}$, or

$$\tilde{\mathbf{r}} = U_T \mathbf{r}^* U_T^{-1} = U_T \mathbf{r} U_T^{-1} = \mathbf{r}, \tag{C.10}$$

$$\tilde{\mathbf{p}} = U_T \mathbf{p}^* U_T^{-1} = -U_T \mathbf{p} U_T^{-1} = -\mathbf{p}. \tag{C.11}$$

In order to satisfy these conditions we do not need additional operators U_T, so it is sufficient to put $U_T = 1$. The $\mathcal{T}$-invariant Hamiltonian has to be an even function of momenta $\mathbf{p}$. For free motion the stationary solution is a plane wave $\exp(i\mathbf{k} \cdot \mathbf{r})$ with (c-number rather than operator) momentum $\mathbf{p} = \hbar\mathbf{k}$. The time reversal operation gives, for $U_T = 1$, the conjugate function $\exp(-i\mathbf{k} \cdot \mathbf{r})$, and this is what we expect for the wave propagating in the reverse direction. The solutions with momenta $\mathbf{p}$ and $-\mathbf{p}$ are degenerate.

C.2 Spin Transformation and Kramer's Theorem

For particles with intrinsic degrees of freedom as spin, it is necessary to specify the unitary matrix U_T that would ensure a correct transformation of these variables. Any angular momentum operator $\mathbf{J}$ is $\mathcal{T}$-odd,

$$\tilde{\mathbf{J}} = \mathcal{T} \mathbf{J} \mathcal{T}^{-1} = U_T \mathbf{J}^* U_T^{-1} = -\mathbf{J}. \tag{C.12}$$

For the orbital part $\mathbf{l}$ this follows from the transformation of the momentum $\mathbf{p}$, (C.11). But we need an extra operator U_T to transform the spin variables.

In the standard representation of the Pauli matrices (A.82), only one of them, σ_y, is imaginary, while σ_x and σ_z are real. This corresponds to the usual choice of phases of the matrix elements of the angular momentum (A.72)–(A.74) when the lowering $J_x - iJ_y$ and raising $J_x + iJ_y$ combinations have real matrix elements (A.71). In this representation one can take

$$U_T = \eta_T \sigma_y \tag{C.13}$$

with an arbitrary phase factor η_T, $|\eta_T|^2 = 1$, as the unitary operator performing time reversal. Using the identity (A.83) accumulating the whole algebra of the Pauli matrices, it is easy to check that

$$\tilde{\mathbf{s}} = U_T \mathbf{s}^* U_T^{-1} = -\mathbf{s}, \tag{C.14}$$

as it should be under time reversal (C.12).

Consider a system of A particles with spin $\frac{1}{2}$. The natural generalization of (C.13) should be

$$U_T = (\eta_T)^A \sigma_y(1) \cdots \sigma_y(A), \tag{C.15}$$

since the spin variables of all particles are to be reversed. Taking into account that the matrices σ_y are imaginary and $\sigma_y^2 = 1$, we find for this system

$$T^2 = U_T \mathcal{K} U_T \mathcal{K} = (-)^A. \tag{C.16}$$

Let a system with a $\mathcal{T}$-invariant Hamiltonian be in a stationary state Ψ. If this state is not degenerate, it can be changed under time reversal by not more than a phase factor. $T\Psi = \exp(i\alpha)\Psi$. But then

$$T^2\Psi = T(e^{i\alpha}\Psi) = e^{-i\alpha}T\psi = e^{-i\alpha}e^{i\alpha}\Psi = \Psi. \tag{C.17}$$

Hence, for a nondegenerate state $T^2 = 1$ regardless of a number of particles. According to (C.16), this means that a system with an odd number of particles with spin $\frac{1}{2}$ cannot have a nondegenerate stationary state. We came to the *Kramers theorem*: stationary states of a $\mathcal{T}$-invariant system of an odd number of particles with spin $\frac{1}{2}$ are *degenerate*, at least twofold.

In the simplest case of a single particle with no spin-orbit coupling or other spin-dependent forces, this is merely a degeneracy of spin states $\chi_\pm$. For a particle in a central field which includes a spin-orbit potential, a stationary single-particle state $|jm\rangle$ with total angular momentum $j = l \pm 1/2$ is $(2j+1)$-degenerate since the rotationally invariant Hamiltonian cannot change its eigenvalue if the orientation is changed. An external *electric* field can split this degenerate multiplet. However, the *electric* field, like any other field keeping time reversal invariance, does not distinguish between the time-conjugate orbits $|jm\rangle$ and $|j-m\rangle$, they stay degenerate. An actual splitting depends on m^2. In cases with no axial symmetry, as in crystals, m is not a constant of motion anymore, but the degeneracy of time conjugate orbits still holds.

Contrary to that, the *magnetic* field changes sign under time reversal. A system with an external magnetic field $\mathbf{B}$ is not time reversal invariant; the level splitting $E_m(\mathbf{B})$ depends on m and the degeneracy is lifted. If a source (current) generating magnetic field **B** is a part of the system under consideration, so that the total time reversal operation includes $\mathbf{B} \to -\mathbf{B}$, the entire system becomes again $\mathcal{T}$-invariant. Then the degeneracy is restored because for each state $|m; \mathbf{B}\rangle$ there is a conjugate state $|-m; -\mathbf{B}\rangle$ with equal energy. If a system is externally cranked, the angular velocity $\mathbf{\Omega}$ also changes sign under time reversal, and the situation is the same as for magnetic field.

C.3 Time-conjugate Orbits

As we saw in the preceding sections, the behavior of the wavefunction under time reversal depends on the spin of the state and on the representation. We will use the representation where the spinors are transformed with the matrix U_T of (C.13) with the phase factor $\eta_T = -i$. Thus, at our choice of η_T, the time reversal operator coincides with the rotation around the y-axis by an angle 180°,

$$U_T = R_y(\pi). \tag{C.18}$$

Acting on the spinor χ_m with $s_z = m = \pm(\frac{1}{2})$, the operator U_T changes $m \to -m$, and the phase factor gives

$$U_T\chi_+ = \chi_-, \quad U_T\chi_- = -\chi_+, \tag{C.19}$$

which can be expressed as

$$U_T\chi_m = (-)^{1/2-m}\chi_{-m}. \tag{C.20}$$

We know that, with respect to rotations, a system with angular momentum J can be thought of as constructed of $2J$ spins $\frac{1}{2}$. Looking at the time reversal behavior, we have, as in the proof of the Kramers theorem, to perform the transformation (C.20) for each spin. As a result, the state $|JM\rangle$ changes the sign of M and acquires the phase factor with the exponent $\sum(\frac{1}{2} - m) = J - M$. Thus, the definition of the time conjugate state, consistent with (C.20), is

$$|\tilde{JM}\rangle = U_T|JM\rangle = (-)^{J-M}|J\,-M\rangle. \tag{C.21}$$

Note that the second time reversal would restore the original state $|JM\rangle$ with the phase factor $(-)^{2J}$ which equals 1 for an integer J and -1 for a half-integer J.

We already witnessed the appearance of the phase (C.21) in the vector coupling of angular momenta when it was related to reversed motion, $\mathbf{J} \to -\mathbf{J}$. The definition (C.21) is consistent with the phase choices of matrix elements of angular momenta and $3j$-symbols. Unfortunately, the traditional definition of spherical functions Y_{lm} differs from the one suggested by (C.21). Since Y_{lm} are functions of the coordinates, they undergo complex conjugation under time reversal, and the related phase factor is $(-)^m$,

$$Y_{lm}(\mathbf{n}) \Rightarrow Y^*_{lm}(\mathbf{n}) = (-)^m Y_{l-m}(\mathbf{n}), \tag{C.22}$$

instead of $(-)^{l-m}$ as it would be in accordance with the rule (C.21), This was the reason for the modified definition of spherical harmonics used by many authors, for example in

Landau and Lifshits [LL65], where the extra factor i^l is added to the normal expression of Y_{lm}. Then the complex conjugation agrees with (C.21) since $(i^l)^* = (-)^l i^l$. One should be careful in using the phase conventions of various authors.

C.4 Two-component Neutrino and Fundamental Symmetries

In the limit of zero mass, the neutrino reveals remarkable properties. For a massless particle, there is no rest frame. In any frame it is moving with speed of light. It has spin $\frac{1}{2}$, and two independent spin states can be classified by taking the momentum axis as that of the quantization. Actually this is the only physical direction associated with a massless particle. Then the spin projection defines the helicity (2.22). A particle with $h = +1 (h = -1)$ is similar to a right (left) screwdriver. These states are analogous to the circularly polarized states of the photon (spin 1 with projections ± 1 onto the direction of the wave propagation).

The combination of definite helicity and absence of mass produces new important consequences. Helicity is a scalar with respect to rotations, but in general it is not a Lorentz scalar. For example, it has no meaning at all in the rest frame. However, for a massless particle, helicity is Lorentz-invariant, the momentum of a particle is transformed together with its spin.

The experiments show that the neutrinos ν ("particles") are always left-polarized, whereas the antineutrinos $\bar{\nu}$ ("antiparticles") are always right-polarized. This statement would be exact in the limit of zero mass; for nonzero mass, the degree of longitudinal polarization is v/c but in the majority of physical situations the neutrino velocity is close to the speed of light. The unique correlation of the *lepton number,* which distinguishes particles from antiparticles, with helicity manifests that some fundamental symmetries are violated in nature.

First of all, parity is not conserved any more. The space inversion changes the sign of the helicity but does not convert the neutrino into the antineutrino. Thus, applying the inversion $\mathcal{P}$ to the left-polarized neutrino, we obtain the nonexisting right-polarized antineutrino—the symmetry with respect to the $\mathcal{P}$ operation is lost. Since the neutrinos are produced in weak interactions only, we conclude that, in contrast to strong and electromagnetic interactions, the weak interactions do not conserve parity. If so, the exact stationary states in the nuclear world, where all interactions are present simultaneously, do not have in general certain parity Π. They are superpositions $\alpha|\Pi\rangle + \beta|-\Pi\rangle$. However, because the admixtures of the opposite parity states are due to the weak interactions, typically one of the coefficients in this combination is very small, except for the cases where some nuclear enhancement mechanisms significantly increase parity mixing.

One of the signatures of the parity nonconservation is the mixed character of the electromagnetic transitions between the two states. Assume, for example, that an unperturbed excited state has quantum numbers $J^{\Pi} = 1^+$, and we observe the magnetic dipole radiation (M1) to a lower state $J^{\Pi} = 0^+$, in agreement with the selection rules for M1-operators:

$\Delta J = 1$, no parity change. The electric dipole (E1) transition between these states, permitted by angular momentum, is forbidden by parity. It becomes allowed because of the admixtures of opposite parity to initial and final states. The corresponding amplitude is proportional to the interference of two components of the wavefunctions, $\alpha_1\beta_2^*$ or $\beta_1\alpha_2^*$.

Another manifestation can be seen in nuclear reactions, when the cross sections of processes interconnected by the inversion transformation turn out to be different. Sometimes it is formulated as the statement that "the results of identical experiments in mirror-reflected laboratories are not mirror-reflected." Thus, in the first experiment where the parity nonconservation was discovered Wu et al. [Wu57], β-decay of polarized nuclei ^{60}Co (here the β electron is accompanied by the electronic antineutrino $\bar{\nu}_e$),

$$^{60}\text{Co} \rightarrow {}^{60}\text{Ni} + e^- + \bar{\nu}_e, \tag{C.23}$$

the angular distribution of the decay electrons, $\sim(1 + a\cos\theta)$, where the angle θ is the one between the electron momentum and the spin direction of the initial nucleus, displayed the preference for the electrons moving opposite to the nuclear spin. This is essentially the quantity of the same type as the helicity (B.40), pseudoscalar that shows that in the mirror-reflected laboratory the angular distribution would be different, $\sim(1 - a\cos\theta)$. An impressive example was seen in scattering of longitudinally polarized slow neutrons off unpolarized heavy nuclei, when the cross sections for different neutron helicities were different (here a very large nuclear enhancement of the effect was observed).

C.5 Charge Conjugation

Another important discrete symmetry is related to the existence of particles and antiparticles. The corresponding transformation $\mathcal{C}$ converts all particles into their antiparticles, changing the signs of all charges (electric, baryonic, leptonic, strangeness, etc.) to the opposite. Neutral particles such as π^0 or photons (they have all charges equal to zero) are transformed into themselves, and, because $\mathcal{C}^2 = 1$, we can distinguish the neutral particles with definite *charge parity* $C = \pm 1$. The strong and electromagnetic interactions are invariant under the $\mathcal{C}$ operation. An electron in an electromagnetic field behaves similarly to a positron in the field of the opposite direction. In order to reveal the symmetry with respect to charge conjugation, we have to invert the sign of the field, that is, to assign to the quantum of the electromagnetic field, the photon, the charge parity $C_\gamma = -1$.

Among many other consequences of invariance under charge conjugation, one can mention the so-called *Furry theorem* in QED: the processes whose only result is the change of an even (odd) photon number to an odd (even) number, are forbidden. Thus, for example, photon-photon interaction processes $2\gamma \rightarrow 3\gamma$ are impossible. The neutral pion decays into two photons, and therefore has positive charge parity.

The neutrino properties discussed earlier show that in weak interactions the charge conjugation symmetry is also broken. There is no longer a full symmetry between particle and antiparticle worlds because the $\mathcal{C}$ operation transforms the left neutrino into the nonexistent (in the limit of zero mass) left antineutrino. Both $\mathcal{P}$ and $\mathcal{C}$ symmetries are destroyed simultaneously.

The picture is invariant with respect to the *combined inversion* $\mathcal{CP}$. It means the conversion of the left neutrino into the right antineutrino. The transition to the antiworld should be accompanied by the mirror reflection; then the results of the corresponding experiments would be the same. $\mathcal{CP}$-invariance is nearly exact. As far as we know, it is violated in rare decays of K-mesons (kaons), and similar B-mesons. The probability of $\mathcal{CP}$-violating decay of neutral kaons is only 0.2% of the probability of normal $\mathcal{CP}$-conserving decay.

The production of neutrinos in the weak interactions shows that $\mathcal{P}$- and $\mathcal{C}$-symmetries separately are completely violated. However, this is just the most distinct manifestation. As we mentioned, parity nonconservation takes place in the nuclear weak processes without a neutrino as well. The direct observation of the corresponding $\mathcal{C}$-violation unfortunately would require experiments with antinuclei.

C.6 Electric Dipole Moment

After we have discussed that the parity conservation is not the universal rule of particle and nuclear interactions, we can return to the question of allowed and forbidden multipoles. The existence of the electric dipole moment in a system with spin $\geq \frac{1}{2}$ is permitted if the restrictions related to the parity conservation are lifted. Of course we still keep the restrictions imposed by the requirements of the rotational invariance.

However, the problem is more complicated. The dipole operator **d** is a polar vector. Its expectation value can be calculated with the aid of the vector model. This gives for the effective dipole operator for a particle of spin $\frac{1}{2}$:

$$\mathbf{d} = \frac{\langle(\mathbf{d}\cdot\mathbf{s})\rangle}{\mathbf{s}^2}\mathbf{s} = \frac{4}{3}\langle(\mathbf{d}\cdot\mathbf{s})\rangle\mathbf{s}. \tag{C.24}$$

The result is determined by the expectation value of the pseudoscalar quantity $(\mathbf{d}\cdot\mathbf{s})$. Since, as a result of the weak interactions, the stationary states have no certain parity, this expectation value can differ from zero. However, a nonzero value of this quantity would contradict time reversal invariance.

Indeed, the spinors $|\frac{1}{2}, m\rangle$ with spin projection m are transformed according to (C.20) under time reversal. The dipole moment **d**, like the coordinate vector **r**, is invariant under $\mathcal{T}$-transformation ("$\mathcal{T}$-even") while the spin vector **s**, like any angular momentum, is $\mathcal{T}$-odd. Therefore the scalar product $(\mathbf{d}\cdot\mathbf{s})$ is $\mathcal{T}$-odd. If $\mathcal{T}$-invariance holds, the expectation value of a time-reversed operator in a time-reversed state should be the same as before the $\mathcal{T}$-transformation,

$$\langle\tfrac{1}{2}, m|(\mathbf{d}\cdot\mathbf{s})|\tfrac{1}{2}, m\rangle = \langle\tfrac{1}{2}, -m| - (\mathbf{d}\cdot\mathbf{s})|\tfrac{1}{2}, -m\rangle^* = -\langle\tfrac{1}{2}, -m|(\mathbf{d}\cdot\mathbf{s})|1/2, -m\rangle \tag{C.25}$$

(expectation values of any Hermitian operator are real). At the same time, the quantity $(\mathbf{d}\cdot\mathbf{s})$ is a rotational scalar, and its expectation value is the same in all substates of the multiplet. Thus, it is equal to zero. This derivation holds for any angular momentum J of the state (not necessarily spin $\frac{1}{2}$). The nonzero helicity $\propto (\mathbf{p} \cdot \mathbf{s})$, in contrast to (C.25), can exist being the product of the two $\mathcal{T}$-odd vectors.

We have shown that a nonzero electric dipole moment of a particle in a stationary state would be a signature of the combination of the parity nonconservation along with the violation of the time reversal invariance. The experiment up to now was unable to discover a dipole moment of a particle. Current data are compatible with zero at the uncertainty level of 10^{-23} e·cm for the proton, 10^{-25} e·cm for the neutron, and 10^{-26} e·cm for the electron.

Recently, the *anapole moment* was discovered experimentally in the nucleus ^{133}Cs. This is a quantity characteristic to the current in the toroidal coil; the main contribution to the anapole moment is due to the operator $\mathbf{a} = [\mathbf{r} \times \mathbf{s}]$. We see that $\mathbf{a}$ is a polar $\mathcal{T}$-odd vector which can exist in nuclear states with nonzero spin J, the corresponding effective operator being

$$\mathbf{a} = \frac{\langle(\mathbf{a} \cdot \mathbf{J})\rangle}{J(J+1)}\mathbf{J}. \tag{C.26}$$

The quantity $(\mathbf{a} \cdot \mathbf{J})$ is a $\mathcal{T}$-even pseudoscalar, and requires only parity nonconservation but not $\mathcal{T}$-violation. The anapole moment was discovered by the parity violation in atomic radiative transitions induced by the weak interactions between atomic electrons and the nucleus.

C.7 $\mathcal{CPT}$-Invariance

We conclude this appendix with brief mention of the famous *CPT*-theorem (R. Lüders and W. Pauli). According to this theorem, any theory preserving the fundamental principles as Lorentz invariance, unitarity (conservation of probability), and the proper relation between the spin value and statistics of particles (integer values correspond to Bose-Einstein statistics, while half-integer spins correspond to Fermi-Dirac statistics) is invariant with respect to the combined application of charge conjugation $\mathcal{C}$, spatial inversion $\mathcal{P}$ and time reversal $\mathcal{T}$. Basically this means that antiparticles in the world obtained by the inversion of all four Minkowski coordinates behave in the same way as particles in the original world.

It follows from the $\mathcal{CPT}$-theorem that particles and their antiparticles have exactly equal masses. If they are unstable, their full lifetimes are also exactly equal (in general, this is not correct for partial lifetimes into specific decay channels). There is no experimental evidence for a violation of the $\mathcal{CPT}$-theorem.

Validity of the $\mathcal{CPT}$-theorem allows one to test fundamental symmetries in an indirect way. For instance, the violation of the combined inversion $\mathcal{CP}$ in the decays of neutral kaons is, according to the $\mathcal{CPT}$-theorem, at the same time a signal of $\mathcal{T}$-noninvariance. Presence in an experiment of $\mathcal{T}$-invariant effects violating parity, as in the case of the anapole moment, proves that the charge symmetry $\mathcal{C}$ is violated as well. This statement can be made with no direct measurements of similar processes with antimatter.