# Appendix D Relativistic Quantum Mechanics

## D.1 Lagrangians

For discrete systems the *Hamilton principle* yields the *Lagrangian equations*

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt = 0 \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad \text{(D.1)}$$

where the *Lagrangian* is the difference between the kinetic energy and the potential energy, i.e. $L = T - V$. The *Hamiltonian* is given by $H = \sum_i p_i \dot{q}_i - L$, where $p_i = \partial L/\partial \dot{q}_i$.

If $\eta_i$ is the displacement of particle $i$ from its equilibrium position (see figure D.1), then

$$L = \frac{1}{2}\sum_i^N \left[m\dot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2\right] = \sum_i^N a\frac{1}{2}\left[\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2\right] = \sum_i^N a\mathcal{L}_i, \qquad \text{(D.2)}$$

where $a$ is the separation distance between the equilibrium positions of two neighboring particles and $\mathcal{L}_i$ is the linear Lagrangian density.

In continuum systems we can make the substitutions

$$a \rightarrow dx, \qquad \frac{m}{a} \rightarrow \mu = \text{linear mass density},$$
$$\frac{\eta_{i+1} - \eta_i}{a} \rightarrow \frac{\partial \eta}{\partial x}, \qquad ka \rightarrow Y = \text{Young modulus}. \qquad \text{(D.3)}$$

Now $\eta$ is a function of $x$; $\eta(x)$, and $L = \int \mathcal{L}\, dx$ is the *Lagrangian density* given by

$$\mathcal{L} = \frac{1}{2}\left[\mu\dot{\eta}_i^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right]. \qquad \text{(D.4)}$$

The variational principle, $\delta \int_{t_1}^{t_2} L\, dt = 0$, (using $\eta(t_2) = \eta(t_1) = 0$) leads to the Euler-Lagrange equations

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial \eta/\partial x)} + \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial \eta/\partial t)} - \frac{\partial \mathcal{L}}{\partial \eta} = 0. \qquad \text{(D.5)}$$

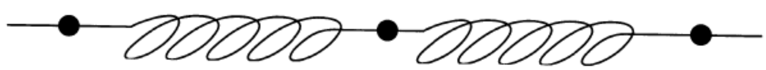

**Figure D.1.** Particles connected by identical springs.

For the example given above, the Euler-Lagrange equations become

$$Y\frac{\partial^2\eta}{\partial x^2} - \mu\frac{\partial^2\eta}{\partial t^2} = 0, \tag{D.6}$$

which is the wave equation with velocity $\sqrt{Y/\mu}$.

The *Hamiltonian density* is defined by

$$\mathcal{H} = \dot{\eta}\frac{\partial\mathcal{L}}{\partial\dot{\eta}} - \mathcal{L} = \frac{1}{2}\mu\dot{\eta}^2 + \frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2 = T + V. \tag{D.7}$$

The quantity $\partial\mathcal{L}/\partial\dot{\eta}$ is known as the canonical momentum.

### D.1.1 Covariance

Generalizing (D.5) to three dimensions, $\mathcal{L}$ depends on $\phi$, $\partial\phi/\partial x_k$ $(k = 1,\ 2,\ 3)$, and $\partial\phi/\partial t$. The Euler-Lagrange equations become

$$\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\frac{\partial\mathcal{L}}{\partial\left(\partial\phi/\partial x_k\right)} + \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left(\partial\phi/\partial t\right)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{D.8}$$

A 4-*vector* is defined by

$$b_\mu = \left(b_0,\ b_1,\ b_2,\ b_3\right) = \left(b_0,\ \mathbf{b}\right) \tag{D.9}$$

with the convention that the Greek letters $\mu,\ \nu,\ \lambda$, etc. vary from 0 to 3, and the roman letters $i, j, k$, etc. vary from 1 to 3.

The coordinate vector $x_\mu$ is defined by

$$x_\mu = \left(x_0,\ x_1,\ x_2,\ x_3\right) = \left(ct,\ \mathbf{x}\right), \tag{D.10}$$

whereas the coordinate vector $x^\mu$ is defined by

$$x^\mu = \left(-x_0,\ x_1,\ x_2,\ x_3\right) = \left(-ct,\ \mathbf{x}\right). \tag{D.11}$$

A Lorentz transformation is given by

$$x'_\mu = a^\nu_\mu x_\nu, \tag{D.12}$$

where a repeated superscript and subscript has the meaning of a sum. Since a Lorentz transformation preserves the length of a vector (that is, $x^\mu x_\mu = x'^\mu x'_\mu$), we have

$$a^\mu_\nu a^\nu_\lambda = \delta^\mu_\lambda, \quad \left(a^{-1}\right)^\mu_\nu = a^\mu_\nu, \tag{D.13}$$

where $\delta_{\nu\lambda}$ is a Kronecker delta. Thus,

$$x_\mu = \left(a^{-1}\right)^\nu_\mu x'_\nu = a^\nu_\mu x'_\nu. \tag{D.14}$$

By definition, a 4-vector transforms like $x_\mu$., and

$$\frac{\partial}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial}{\partial x_\nu} = a^\nu_\mu \frac{\partial}{\partial x_\nu}. \tag{D.15}$$

Thus, $\partial/\partial x_\mu$ is also a 4-vector.

A *scalar product* is defined by

$$b \cdot c = b^\mu c_\mu = \mathbf{b}\cdot\mathbf{c} - b_0 c_0 \tag{D.16}$$

and does not change in a Lorentz transformation

$$b' \cdot c' = a^\nu_\mu b_\nu a^\lambda_\mu c_\lambda = \delta^{\nu\lambda} b_\nu c_\lambda = b \cdot c. \tag{D.17}$$

A *second degree tensor* transforms like

$$t'_{\mu\nu} = a^\lambda_\mu a^\sigma_\nu t_{\lambda\sigma}, \tag{D.18}$$

and similarly for tensors of higher dimension.

Equation (D.8) can be written as

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\partial\phi/\partial x_\mu\right)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{D.19}$$

This equation is covariant, that is, it has the same form in all systems of reference.

## D.2 Electromagnetic Field

The *Maxwell equations*, in Heaviside-Lorentz units, are given by

$$\nabla\cdot\mathbf{E} = \rho, \qquad \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{\mathbf{j}}{c}, \tag{D.20}$$

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0. \tag{D.21}$$

In these units the fine-structure constant is $e^2/4\pi\hbar c \simeq 1/137.04$, which is equal to $e^2/\hbar c$ in the Gaussian system (cgs) and $e^2/\left(4\pi\hbar c\epsilon_0\right)$ in MKS units. The fields and potentials in these units are related to the corresponding quantities in the Gaussian system by $1/\sqrt{4\pi}$; for example, $\frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)$ in these units must read $(1/8\pi)\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)$ in Gaussian units. However, expressions like $\mathbf{p} - e\mathbf{A}/c$ are the same in both units, because

$$\left(\sqrt{4\pi}e\right)\left(\mathbf{A}/\sqrt{4\pi}\right) = e\mathbf{A}. \tag{D.22}$$

Introducing the *antisymmetric* tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}, \tag{D.23}$$

and the current

$$j_\mu = (c\rho,\ \mathbf{j})\,, \tag{D.24}$$

(D.21) can be written in the compact form

$$\frac{\partial F\mu\nu}{\partial x_\nu} = \frac{j_\mu}{c}. \tag{D.25}$$

Because $F_{\mu\nu}$ is antisymmetric, it obeys the relation $\partial F_{\mu\nu}/\partial x_\mu \partial x_\nu = 0$, which means that

$$\frac{\partial j_\mu}{\partial x_\mu} = 0. \tag{D.26}$$

The *vector potential* is introduced by

$$\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = F_{\mu\nu}, \tag{D.27}$$

and (D.20) can be written as

$$t_{\lambda\mu,\nu} + t_{\mu\nu,\lambda} + t_{\nu\lambda,\mu} = 0, \tag{D.28}$$

where the third degree tensor $t_{\lambda\mu,\nu}$ is defined by

$$t_{\lambda\mu,\nu} = \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\mu}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_\mu}\right). \tag{D.29}$$

Using the Euler-Lagrange equations, it is straightforward to show that the Lagrangian

$$\mathcal{L} \;= -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \left(j_\mu A_\mu\right)/c, \tag{D.30}$$

reproduces the Maxwell equations (D.20) and (D.21).

We can rewrite (D.25) as

$$\Box A_\mu - \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu}{\partial x_\nu}\right) = -\frac{j_\mu}{c}. \tag{D.31}$$

We can redefine $A_\mu$, without changing $F_{\mu\nu}$, as

$$A_\mu^{\text{new}} = A_\mu^{\text{old}} + \frac{\partial \chi}{\partial x_\mu}, \quad \text{and} \tag{D.32}$$

where

$$\Box\chi = -\frac{\partial A_\mu^{\text{old}}}{\partial x_\mu}, \tag{D.33}$$

and thus,

$$\frac{\partial A_\mu^{\text{new}}}{\partial x_\mu} = \frac{\partial A_\mu^{\text{old}}}{\partial x_\mu} + \Box\chi = 0. \tag{D.34}$$

Since the vector potential is used to simplify the calculations, we can therefore use a simpler equation

$$\Box A_\mu = -\frac{j_\mu}{c}, \tag{D.35}$$

where $A_\mu$ obeys

$$\frac{\partial A_\mu}{\partial x_\mu} = 0. \tag{D.36}$$

Equation (D.36) is known as *Lorentz condition.*

But, even by using (D.36), the potential $A_\mu$ is not univocally determined. We can still make an additional transformation of the form

$$A_\mu \to A'_\mu = A_\mu + \frac{\partial \Lambda}{\partial x_\mu}, \tag{D.37}$$

where $\Lambda$ obeys the equation

$$\Box \Lambda = 0. \tag{D.38}$$

The transformation (D.37) is known as the *gauge transformation.*

## D.3 Relativistic Equations

The energy momentum relation

$$E^2 = p^2 + m^2 \Longrightarrow \widehat{H}_0 \psi(\mathbf{x}, t) = \left[\widehat{\mathbf{p}}^2 + m^2\right] \psi(\mathbf{x}, t), \tag{D.39}$$

together with the quantization rules $\widehat{\mathbf{p}} = -i\nabla$ and $\widehat{H}_0 = i\partial_t$, leads to the wave equation

$$\left[\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right] \psi(\mathbf{x}, t) = 0. \tag{D.40}$$

This is known as the *Klein-Gordon equation.* It is considered as the appropriate equation for spin-zero particles, for example, the $\pi$-mesons.

Another relativistic equation, proposed by Dirac, is linear in the space-time derivatives:

$$i\frac{\partial \Psi}{\partial t}(\mathbf{x}, t) = H_0 \Psi(\mathbf{x}, t). \tag{D.41}$$

where

$$H_0 = \frac{1}{i}\left[\alpha_2 \frac{\partial}{\partial x^1} + \alpha_2 \frac{\partial}{\partial x^2} + \alpha_3 \frac{\partial}{\partial x^3}\right] + \beta m. \tag{D.42}$$

Above, $\alpha_i$ and $\beta$ are dimensionless constants, commuting with $\mathbf{r}$ and $\mathbf{p}$. Defining

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \tag{D.43}$$

we get

$$H_0 = \alpha \cdot \mathbf{p} + \beta m. \tag{D.44}$$

Applying the operator $\partial/\partial t$ in (D.41) gives

$$\frac{\partial}{\partial t}\left(i\frac{\partial \Psi}{\partial t}\right) = (\alpha \cdot \mathbf{p} + \beta m)\frac{\partial \Psi}{\partial t}$$

$$\Longrightarrow i\frac{\partial^2 \Psi}{\partial t^2} = (\alpha \cdot \mathbf{p} + \beta m)\left(\alpha \cdot \frac{\mathbf{p}}{i}\Psi + \frac{\beta}{i} m \Psi\right).$$

$$= i\sum_{kj} \frac{\alpha_j\alpha_k + \alpha_k\alpha_j}{2} \frac{\partial^2 \Psi}{\partial x^k \partial x^j} - m\sum_k (\alpha_k\beta + \beta\alpha_k)\frac{\partial \Psi}{\partial x^k} - i\beta^2 m^2 \Psi. \tag{D.45}$$

To reduce this equation to the Klein-Gordon equation (for which $E^2 = p^2 + m^2$), it is necessary that

$$\begin{aligned} \alpha_j\alpha_k + \alpha_k\alpha_j &\equiv \{\alpha_k, \alpha_j\} = 2\delta_{kj}, \\ \alpha_k\beta + \beta\alpha_k &\equiv \{\alpha_k, \beta\} = 0, \\ \Longrightarrow \alpha_i^2 &= \beta^2 = 1. \end{aligned} \tag{D.46}$$

These equations can only be satisfied if $\alpha_i$ and $\beta$ are matrices. Thus $\Psi$ must be a vector with $N$ components:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_N \end{pmatrix}. \tag{D.47}$$

The matrices $\alpha_i$ and $\beta$ must have the properties

1. Hermiticity:

   $$\alpha_i^\dagger = \alpha_i, \quad \beta^\dagger = \beta. \tag{D.48}$$

   because $H_0$ is Hermitian.. This, together with (D.46), implies that the eigenvectors of $\alpha_i$ and $\beta$ must be $\pm 1$.
2. $\mathrm{Tr}\alpha = \mathrm{Tr}\beta = 0$.
3. $N$ must have even dimension.
   Both properties 3 and 4 follow directly by using (D.46).
4. $N \geq 4$.
   $N = 2$ is not possible because the Pauli matrices $(\sigma, I)$ form a complete set of $2 \times 2$ matrices. However, the matrix $I$ always commutes and (D.46) cannot be satisfied. Thus, $N = 4$ is the smallest possibility, and

   $$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \Psi^\dagger = \left(\psi_1^*\psi_2^*\psi_3^*\psi_4^*\right). \tag{D.49}$$

5. Representation of $\alpha_i$ and $\beta$:
   The above conditions lead to many possible representations of $\alpha_i$ and $\beta$. The most popular representation is

   $$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{D.50}$$

   where $\sigma_i$, 1, and 0 in (D.50) are $2 \times 2$ matrices.

### D.3.1 Particle at rest

For a particle at rest $\mathbf{p} = 0$ so that $\nabla\Psi = 0$ and

$$i\frac{\partial\Phi}{\partial t}(\mathbf{x},t) = \beta m\Phi(\mathbf{x},t)\,. \tag{D.51}$$

In terms of the wave function components

$$i\begin{pmatrix} \partial\phi_1/\partial t \\ \partial\phi_2/\partial t \\ \partial\phi_3/\partial t \\ \partial\phi_4/\partial t \end{pmatrix} = m\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m\begin{pmatrix} \phi_1 \\ \phi_2 \\ -\phi_3 \\ -\phi_4 \end{pmatrix}. \tag{D.52}$$

These equations have 4 solutions given by

$$\Phi_1 = e^{-imt}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Phi_2 = e^{-imt}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{D.53}$$

$$\Phi_3 = e^{+imt}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Phi_4 = e^{+imt}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{D.54}$$

We identify $\Phi_1$ and $\Phi_2$ as positive energy solutions for up and down spin states, respectively. The other solutions, $\Phi_3$ and $\Phi_4$, are negative energy solutions, or antiparticle solutions, with spin up and down, respectively. We therefore see the necessity of 4 components.

### D.3.2 Covariant form: $\gamma$ matrices

Defining the matrices

$$\gamma^0 \equiv \beta, \quad \gamma^i \equiv \beta\alpha_i, \quad \boldsymbol{\gamma} \equiv \beta\boldsymbol{\alpha}, \quad \gamma^\mu \equiv \left(\gamma^0, \boldsymbol{\gamma}\right), \tag{D.55}$$

multiplying the Dirac equation by $\gamma^0$, and using $\beta^2 = 1$, we obtain

$$\begin{aligned} &\left(i\gamma^0\frac{\partial}{\partial t} - \boldsymbol{\gamma}\cdot\frac{\nabla}{i}\right)\Psi(\mathbf{x},t) = m\Psi(\mathbf{x},t)\,, \\ &\quad\left(\gamma^0 p_0 - \boldsymbol{\gamma}\cdot\mathbf{p}\right)\Psi(\mathbf{x},t) = m\Psi(\mathbf{x},t)\,. \end{aligned} \tag{D.56}$$

or

$$\gamma^\mu p_\mu \Psi(\mathbf{x},t) = m\Psi(\mathbf{x},t)\,,$$

where $p_\mu = i\partial_\mu$.

**Table D.1: Bilinear covariants built with Dirac $\gamma$ matrices.**

| $\Gamma_{4\times4}$ | Definition | Transformation | Number |
|---|---|---|---|
| $\gamma$ | $\beta\alpha$ | space vector | 3 |
| $\gamma_0$ | $\beta$ | time vector | 1 |
| $1 \equiv I$ | identity | scalar | 1 |
| $\sigma^{\mu\nu}$ | $\frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$ | traceless tensor | 6 |
| $\gamma_5$ | $i\gamma^0\gamma^1\gamma^2\gamma^3$ | pseudoscalar | 1 |
| $\gamma_5\gamma^\mu$ | $\gamma_5\gamma^\mu$ | pseudovector | 4 |

Using (D.46) we can prove that

$$\begin{aligned} &\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu \equiv \left\{\gamma^\nu, \gamma^\mu\right\} = 2g^{\mu\nu}, \\ &\quad \left(\gamma^i\right)^2 = -1 \qquad (i = 1, 2, 3), \quad \left(\gamma^0\right)^2 = 1, \\ &\qquad \gamma^\dagger = -\gamma, \quad \gamma_0^\dagger = \gamma_0. \end{aligned} \tag{D.57}$$

The most popular representations of $\gamma$ are

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{D.58}$$

It is also useful to generalize the matrices $\sigma$ as

$$\sigma^{ij} = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad \sigma^{0i} = i\alpha_i = i \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}. \tag{D.59}$$

The indices $(i, j, k)$ take values 1, 2, and 3 (or $x$, $y$, $z$), and can be cyclically permuted. The $\sigma$'s are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{D.60}$$

There are only 16 $4 \times 4$ independent matrices. Using the notation $\Gamma_i$ for these matrices, one can show that they can all be built from the $\gamma$ matrices, as shown in table D.1.

The table also shows the transformation properties of $\overline{\Psi}\Gamma\Psi$. The spatial parts $\sigma^{ij}$ $(i, j = 1, 2, 3)$ are related to spin, while the mixed space-time parts $\sigma^{0\mu}, \sigma^{\mu 0}$ are related to the velocities (proportional to $\alpha$). To emphasize these relations, we write

$$\Sigma_k = \sigma^{ij}_{4\times4}, \tag{D.61}$$

where $\Sigma$ is seen as a four-dimensional generalization of the Pauli matrices.

## D.4 Probability and Current

Multiplying the Dirac equation to the left by $\Psi^\dagger$ and subtracting the result by the Hermitian conjugate of this operation we get

$$\frac{\partial\left(\Psi^\dagger\Psi\right)}{\partial t}+\nabla\cdot\left(\Psi^\dagger\alpha\Psi\right)=0. \tag{D.62}$$

This equation has the form of a continuity equation, where the probability is given by

$$\rho=\Psi^\dagger\Psi=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2, \tag{D.63}$$

and the current is given by

$$\mathbf{j}=\Psi^\dagger\alpha\Psi\equiv\Psi^\dagger\beta\beta\alpha\Psi\equiv\overline{\Psi}\gamma\Psi. \tag{D.64}$$

Note that $\mathbf{j}$ is given in terms of the *Dirac adjoint* $\overline{\Psi}$ defined by

$$\overline{\Psi}=\Psi^\dagger\beta\equiv\Psi^\dagger\gamma_0. \tag{D.65}$$

The continuity equation can be expressed as $\partial_\mu j^\mu=0$, where

$$j^\mu=(\rho,\mathbf{j})=\overline{\Psi}\gamma^\mu\Psi. \tag{D.66}$$

## D.5 Wavefunction Transformation

The Lorentz transformations are given by

$$x'^\nu=a^\nu_\mu x^\mu. \tag{D.67}$$

In a new referential $O'$, $\Psi'\left(x'\right)$, is related to $\Psi(x)$ in the referential $O$, by

$$\Psi'\left(x'\right)=L_\nu(a)\,\Psi(x), \tag{D.68}$$

where $L_\nu(a)$ changes only the components of $\Psi(x)$. The Dirac equation must be invariant under this transformation. That is,

$$\begin{aligned} i\gamma^\mu\frac{\partial\Psi(x)}{\partial x^\mu}&=m\Psi(x),\\ i\gamma^\mu\frac{\partial\Psi'\left(x'\right)}{\partial x'^\mu}&=m\Psi'\left(x'\right). \end{aligned} \tag{D.69}$$

Using

$$\Psi(x)=L_\nu^{-1}(a)\,\Psi'\left(x'\right),\qquad \frac{\partial}{\partial x^\mu}=a^\nu_\mu\frac{\partial}{\partial x'^\nu}, \tag{D.70}$$

in (D.69), we get

$$i\gamma^\mu a^\nu_\mu\frac{\partial}{\partial x'^\nu}L_\nu^{-1}(a)\,\Psi'\left(x'\right)=mL_\nu^{-1}(a)\,\Psi'\left(x'\right). \tag{D.71}$$

Multiplying the equation above by $L_\nu(a)$, and because $L_\nu(a)$ commutes with the derivatives and with $a^\nu_\mu$, one gets

$$\left[a^\nu_\mu L_\nu(a)\,\gamma^\mu L_\nu^{-1}(a)\right] i\frac{\partial}{\partial x'^\nu}\Psi'(x') = m\Psi'(x'). \tag{D.72}$$

Therefore, Lorentz invariance implies

$$\left[a^\nu_\mu L_\nu(a)\,\gamma^\mu L_\nu^{-1}(a)\right] = \gamma^\mu,$$
$$\Longrightarrow a^\nu_\mu \gamma^\mu = L_\nu^{-1}(a)\,\gamma^\mu L_\nu(a). \tag{D.73}$$

For a spin $\frac{1}{2}$ particle, the rotation by an angle $\theta$ implies that

$$\Psi'(x') = U_R(\theta)\,\Psi(x), \quad \text{where} \quad U_R(\theta) = e^{-i\mathbf{J}\cdot\theta} = e^{-i\theta\cdot\sigma/2}, \tag{D.74}$$

where $\sigma/2$ is the generator of infinitesimal rotations. Since the generalization of the Pauli matrices $\sigma_k$ is given by equation (D.61), we assume that the relativistic operator for rotations in 3-dimensions, around the axis $k$, is in the form

$$U_R(\theta) = e^{-i\Sigma_k\theta_k/2}. \tag{D.75}$$

To generalize (D.75) for Lorentz transformations, we imagine these transformations as rotation in space-time and we replace $\sigma^{ij}$, the operator of infinitesimal rotations around the axis $k$, by $\sigma^{0k}$, the generator of infinitesimal velocity transformations along an axis $k$. We see that the rotation "angle" is imaginary. For transformations along an axis, there is a *rapidity parameter* $\lambda$ that determines the velocity $u = u(\lambda)$ and is additive for successive transformations:

$$\cosh\lambda = \gamma_u = \left(1-\beta_u^2\right)^{-1/2} = \left(1-u^2\right)^{-1/2}, \quad \sinh\lambda = \gamma_u\beta_u. \tag{D.76}$$

In terms of this quantity, the Lorentz transformation along the $x$-axis, can be expressed in terms of an imaginary angle:

$$x' = x\cosh\lambda - t\sinh\lambda, \quad \text{and} \quad t' = t\cosh\lambda - x\sinh\lambda. \tag{D.77}$$

An observer $O$ who sees a particle moving with velocity $v$ along $x$ uses (D.77) to determine the *rapidity* of the particle and denotes it by $\lambda_0$. Similarly, the observer $O'$ sees the same particle moving with velocity $v'$ and uses (D.77) to determine the *rapidity*, denoting it by $\lambda_0'$. We can show that the relation between the two velocities is

$$v'(\lambda_0') = \frac{u(\lambda) + v(\lambda_0)}{1+uv'}, \quad \text{that is,} \quad \lambda_0' = \lambda_0 + \lambda. \tag{D.78}$$

Thus, the parameter $\lambda$ is additive for transformations along an axis (as with rotations around an axis). The transformation is "active," and since the conventional Lorentz transformation (D.67), (D.68) is passive, we have a sign to worry about. We assume that the generalization of (D.75) is

$$L_\nu(a) \equiv L_\nu(\lambda) = \exp\left[\mp\frac{1}{2}i\lambda_k\sigma^{0k}\right], \tag{D.79}$$

where $k$ is the axis for the velocity increase and the $+(-)$ sign is for active (passive) transformations. Expanding the potential in a series and using the relation

$$\left(\sigma^{0k}\right)^2 = \left(i\alpha_k\right)^2 = -I_{4\times4}, \tag{D.80}$$

we obtain the relation

$$L_v = \exp\left[\mp\frac{1}{2}i\lambda_k\sigma^{0k}\right] = I\cosh\frac{\lambda_k}{2} \mp i\sigma^{0k}\sinh\frac{\lambda_k}{2}. \tag{D.81}$$

One can show that (D.73) works for $\nu = 0, k$, from the relations

$$\begin{aligned}
&\left\{\gamma^0, \sigma^{0k}\right\} = \left\{\gamma^k, \sigma^{0k}\right\} = 0,\\
&L_v^{-1}\gamma^0 L_v = e^{-i\lambda_k\sigma^{0k}}\gamma^0 = \cosh\lambda_k\gamma^0 - \sinh\lambda_k\gamma^k\\
&\qquad = a^0_\mu\gamma^\mu = \gamma_v\gamma^0 - \gamma_v\beta_v\gamma^k.\\
&L_v^{-1}\gamma^k L_v = \gamma_v\gamma^k - \gamma_v\beta_v\gamma^0.
\end{aligned} \tag{D.82}$$

Note that $L_v$ *is not unitary*, $L_v \neq L_v^{-1}$. This means that the normalization of $\Psi$ changes under the transformation. However, $L_v^\dagger = L_v$ . This change in $\Psi$ is necessary to keep the total probability constant, since a volume element also changes udder this transformation. Although $L_v^\dagger$ is not equal to $L_v^{-1}$, there is a simple relation between them:

$$L_v^{-1} = \gamma_0 L_v^\dagger \gamma_0. \tag{D.83}$$

### D.5.1 Bilinear covariants

In table D.1 we have 16 matrices that, when sandwiched between the Dirac spinor $\Psi$ and its adjoint $\overline{\Psi}$, transform as indicated. For example, the current

$$j^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x) = \Psi^\dagger(x)\gamma^0\gamma^\mu\Psi(x) \tag{D.84}$$

is a 4-vector. For example, relating $j'^\mu(x')$ to $j^\mu(x)$, we have

$$\begin{aligned}
j'^\mu(x') &= \Psi'^\dagger(x')\gamma^0\gamma^\mu\Psi'(x') = \Psi^\dagger(x)L_v^\dagger\gamma^0\gamma^\mu L_v\Psi(x)\\
&= \Psi^\dagger(x)\gamma^0\left(\gamma^0 L_v^\dagger\gamma^0\right)\gamma^\mu L_v\Psi(x) = \Psi^\dagger(x)\gamma^0\left[L_v^{-1}\gamma^\mu L_v\right]\Psi(x)\\
&= \Psi^\dagger(x)\gamma^0\left[a^\mu_\nu\gamma^\nu\right]\Psi(x) = a^\mu_\nu\overline{\Psi}(x)\gamma^\nu\Psi(x),
\end{aligned} \tag{D.85}$$

or

$$j'^\mu(x') = a^\mu_\nu j^\nu(x). \tag{D.86}$$

Similarly, $\overline{\Psi}(x)I\Psi(x)$ is a scalar

$$\overline{\Psi}I\Psi = \overline{\Psi}'I\Psi, \tag{D.87}$$

and $\overline{\Psi}\sigma^{\mu\nu}\Psi$ is second degree tensor

$$\overline{\Psi}'(x')\sigma^{\mu\nu}\Psi'(x') = a^\mu_\alpha a^\nu_\beta\overline{\Psi}(x)\sigma^{\alpha\beta}\Psi(x). \tag{D.88}$$

### D.5.2 Parity

The parity transform is defined by

$$P\Psi(x) = \Psi'(x') = \Psi'(-\mathbf{x}, t). \tag{D.89}$$

To include parity as part of the Lorentz group operators, we have to find a $4 \times 4$ $P$ matrix with the property

$$P^{-1}\gamma^{\nu} P = a\,(P)^{\nu}_{\mu}\,\gamma^{\mu}, \qquad \text{where} a\,(P) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{D.90}$$

Note that this is an improper Lorentz transformation, since $\det(a) = -1$, while the previous transformations were proper, with $\det(a) = 1$. When a matrix $P$ satisfying (D.89) acts on a spinor $\Psi$, the new $P\Psi$ has opposite *intrinsic parity*, that is, $P$ changes the internal parts of the wavefunctions. The parity operator acting on the external part is different and can have its own eigenvalues. For example, for the eigenstates of angular momentum, the parity eigenvalues are the familiar $(-1)^l$.

The choice $P = \gamma^0 = \beta$ for the parity operator has the desired effect:

$$P \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \beta \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix}. \tag{D.91}$$

This relation is the microscopic explanation for why fermions and antifermions have *opposite intrinsic parity* in the Dirac theory.

The *pseudoscalar object* $\gamma^5$ and the *pseudovector* (or *axial vector*) $\gamma^5\gamma^\mu$ of table D.1 behave as a scalar a pseudoscalar under proper Lorentz transformations $[\det(a) = 1]$, but gain a negative sign for parity transformations.

$$P^{-1}\gamma^5 P = -\gamma^5, \quad P^{-1}\gamma^5\gamma^\mu P = -\gamma^5\gamma^\mu. \tag{D.92}$$

## D.6 Plane Waves

In the Dirac theory, the absence of a positive energy solution is identified as an antiparticle. Thus, it is conventional to revert to the spin in the negative energy eigenvectors so that it refers to real antiparticles. For the particle at rest, this means

$$\begin{aligned} &\Phi_1 = \Phi^{(+)}_{0\uparrow}\ (0) = e^{-imt} u^{(+)}_{\uparrow}\ (0), \qquad &&\Phi_2 = \Phi^{(+)}_{0\downarrow} = e^{-imt} u^{(+)}_{\downarrow}\ (0), \\ &\Phi_3 = \Phi^{(-)}_{0\downarrow} = e^{+imt} u^{(-)}_{\downarrow}\ (0), &&\Phi_4 = \Phi^{(-)}_{0\uparrow} = e^{+imt} u^{(-)}_{\uparrow}\ (0), \end{aligned} \tag{D.93}$$

with

$$u_{\uparrow}^{(+)}(0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ u_{\downarrow}^{(+)}(0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ u_{\downarrow}^{(-)}(0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ u_{\uparrow}^{(-)}(0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{D.94}$$

The index 0 refers to the momentum, $m_s$ to spin, and the argument (0) to the position **x**. It is important to remember that the spinors for a particle at rest are base vectors in which a general $4\times 1$ Dirac spinor can be expanded.

For $p\neq 0$ the Dirac equation is

$$\left(i\gamma^0\frac{\partial}{\partial t}+i\gamma.\nabla\right)\Phi(\mathbf{x},t)=m\Phi(\mathbf{x},t). \tag{D.95}$$

The factor $mt$ in the exponent of (D.93) is the scalar product $p^\mu x_\mu$ in the referential where the particle is at rest. Using the covariance property, in another referential one has

$$\Phi_{ps}^{(\pm)}(\mathbf{x},t)=e^{\mp ip^\mu x_\mu}u_s^{(\pm)}(p). \tag{D.96}$$

We obtain the plane wave spinors $u_s^{(\pm)}(p)$ doing a Lorentz transformation along the $z$-axis, with the operator $L_\nu$:

$$\begin{aligned}&u_s^{(\pm)}(p)=L_\nu(\lambda)\,u_s^{(\pm)}(0),\\&L_\nu(\lambda)=e^{\mp i\lambda\sigma^{0k}/2}=I\cosh\frac{\lambda}{2}\mp i\sigma^{0k}\sinh\frac{\lambda}{2}.\end{aligned} \tag{D.97}$$

One can show that for a particle with momentum **p**,

$$\tanh\frac{\lambda}{2}=\frac{p}{E_p+m},\qquad \cosh\frac{\lambda}{2}=\sqrt{\frac{E_p+m}{2m}}, \tag{D.98}$$

and $L_\nu(\lambda)$ can be rewritten as

$$L_\nu(\lambda)=\sqrt{\frac{E_p+m}{2m}}\left(1+\frac{\alpha\cdot\mathbf{p}}{E_p+m}\right). \tag{D.99}$$

Replacing the explicit representation of the matrix $\alpha$ in eq. D.97, we get

$$u_{\uparrow}^{(+)}(p)=N\begin{pmatrix}1\\0\\\frac{p_z}{E_p+m}\\\frac{p_+}{E_p+m}\end{pmatrix},\quad u_{\downarrow}^{(+)}(p)=N\begin{pmatrix}0\\1\\\frac{p_-}{E_p+m}\\-\frac{p_z}{E_p+m}\end{pmatrix}, \tag{D.100}$$

$$u_{\downarrow}^{(-)}(p)=N\begin{pmatrix}\frac{p_z}{E_p+m}\\\frac{p_+}{E_p+m}\\1\\0\end{pmatrix},\quad u_{\uparrow}^{(-)}(p)=N\begin{pmatrix}\frac{p_-}{E_p+m}\\-\frac{p_z}{E_p+m}\\0\\1\end{pmatrix}, \tag{D.101}$$

where

$$p_\pm=p_x\pm ip_y,\quad N=\sqrt{\frac{E_p+m}{2m}}. \tag{D.102}$$

These spinors $u_s^{(\pm)}(p)$ describe free particles with spin $s = \pm\frac{1}{2}$ (in their rest frame), energy $\pm E_p$, and 4-momentum $p$. (Note that the negative energy solutions have their 3-momenta reversed, according to the interpretation for the antiparticle solutions).

### D.6.1 Summary of plane wave spinor properties

1. The normalized wavefunction is

$$\Phi_{ps}^{(\pm)}(\mathbf{x}, t) = e^{\mp i p^\mu x_\mu} u_s^{(\pm)}(p). \tag{D.103}$$

The space-time dependence is in the exponential, the spin dependence is in the spinor, and $p$ *is not* an operator.

2. The spin is a *good quantum number* only in the particle's rest frame or for a motion along $z$. If $p_x = p_y = 0$, while $p_z \neq 0$, the $u$'s are eigenstates of $\Sigma_x$, but with contributions from the small components.
3. The helicity $\sigma \cdot \widehat{\mathbf{p}}$ is a good quantum number for these $u$'s.
4. When (D.103) is inserted in the Dirac equation one obtains the equation for the free spinors $u^{(\pm)}(p)$:

$$\left(\gamma^\mu p_\mu \mp m\right) u^{(\pm)}(p) = 0. \tag{D.104}$$

Note again that $p$ *is not* an operator, and that the equation above is a matricial equation.

5. The explicit Dirac equations for the positive and enervative energies are:

$$\begin{aligned} \left[\gamma^0 p_0 - \gamma \cdot \mathbf{p} - m\right] u^{(+)}(p) &= 0, \\ \left[-\gamma^0 p_0 - \gamma \cdot (-\mathbf{p}) - m\right] u^{(-)}(p) &= 0. \end{aligned} \tag{D.105}$$

We see that the spinors of positive energy have energy and momentum opposite to the spinors of negative energy.

6. The adjoint Dirac spinor $\overline{u}(p) = u^\dagger(p)\gamma^0$ satisfies the transposed Dirac equation:

$$\overline{u}^{(\pm)}(p)\left(\gamma^\mu p_\mu \mp m\right) = 0. \tag{D.106}$$

7. The $u$'s satisfy the Lorentz invariant orthogonality relations:

$$\overline{u}_s^{(b)}(p) u_{s'}^{(b')}(p) = b\delta_{bb'}\delta_{ss'}, \tag{D.107}$$

where $b = \pm$ for the positive and negative energy solutions.

8. The probability density $\rho$ for the plane waves is

$$\rho = \Phi^\dagger \Phi = u_s^{(b)}(p)^\dagger u_{s'}^{(b')}(p) = \frac{E_p}{m}\delta_{bb'}\delta_{ss'}. \tag{D.108}$$

Since $\rho$ is proportional to the energy, it is not a Lorentz invariant.

9. The *completeness relation* is

$$\sum_{b=\pm,\, s=\uparrow\downarrow} b u_s^{(b)}(p)\overline{u}_s^{(b)}(p) = I_{4\times 4}. \tag{D.109}$$

Thus, the mathematical completeness requires negative energy degrees of freedom even for low-energy processes.

10. The *current*

$$\mathbf{j} = \overline{u}_s^{(b)}(p)\boldsymbol{\gamma} u_{s'}^{(b')}(p) = \frac{\mathbf{p}}{m}\delta_{bb'}\delta_{ss'} = \frac{\mathbf{p}}{E_p}\rho\delta_{bb'}\delta_{ss'}, \tag{D.110}$$

has the expected plane wave property, that is, $j = \mathbf{v}_g\rho$, a group velocity times a density.

### D.6.2 Projection operators

The operators

$$\Lambda_\pm(p) = \frac{\pm\gamma^\mu p_\mu + m}{2m} \tag{D.111}$$

have the properties

$$\begin{aligned} \Lambda_+ u(p) &= u(p), \qquad \Lambda_- u(p) = 0, \\ \Lambda_- v(p) &= v(p), \qquad \Lambda_+ v(p) = 0, \\ \Lambda_\pm^2 &= \Lambda_\pm, \qquad \Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0, \\ \Lambda_+ + \Lambda_- &= I, \end{aligned} \tag{D.112}$$

where we define the particle and antiparticle spinors as

$$u_s(p) = u_s^{(+)}(p), \qquad v_s(p) = u_s^{(-)}(p). \tag{D.113}$$

Using the $u$'s and the $v$'s as base vectors, we see the $\Lambda_+$ ($\Lambda_-$) is a *projection operator* that eliminates the antiparticle (particle) part of the wavefunction, filtering the particle (antiparticle) part. Since the basis is complete, we have

$$\sum_i \mid i\rangle\langle i \mid = I, \Rightarrow \sum_i \left[u_s(p)\overline{u}_s(p) - v_s(p)\overline{v}_s(p)\right] = I, \tag{D.114}$$

which can be easily verified by substitution.

From the equations above, we get

$$\Lambda_+(p) = \sum_{s=1}^{2} u_s(p)\overline{u}_s(p) = \frac{\gamma^\mu p_\mu + m}{2m}, \tag{D.115}$$

$$\Lambda_-(p) = \sum_{s=1}^{2} v_s(p)\overline{v}_s(p) = \frac{-\gamma^\mu p_\mu + m}{2m}. \tag{D.116}$$

## D.7 Plane Wave Expansion

One can show that $\sqrt{m/(VE_p)}\Phi_s^{(b)}$ is normalized to unity, showing that the plane wave satisfies the orthogonality conditions:

$$\int d^3x \left[\Phi_s^{(b)}(x)\right]^\dagger \Phi_{s'}^{(b')}(x) = \delta_{bb'}\delta_{ss'}\frac{E_p V}{m}. \tag{D.117}$$

Thus, a general solution of the general Dirac equation has the expansion

$$\Psi(\mathbf{x},t) = \sum_{ps}\sqrt{\frac{m}{VE_p}}\left[e^{i(\mathbf{p}\cdot\mathbf{x}-p_0t)}u_s^{(+)}(p)b_{ps}^{(+)}(\mathbf{p}) + e^{-i(\mathbf{p}\cdot\mathbf{x}-p_0t)}u_s^{(-)}(p)b_{ps}^{(-)}(\mathbf{p})\right]$$
$$= \sum_{ps}\sqrt{\frac{m}{VE_p}}\left[e^{ipx}u_s^{(+)}(p)b_{ps}^{(+)}(\mathbf{p}) + e^{ipx}u_s^{(-)}(p)b_{ps}^{(-)}(\mathbf{p})\right], \tag{D.118}$$

where the $b's$ are expansion coefficients.

## D.8 Electromagnetic Interaction

The interaction with the electromagnetic field is obtained by using the minimal coupling; $p^\mu \to p^\mu - qA^\mu$. The Dirac Hamiltonian becomes

$$\left(i\frac{\partial}{\partial t} - q\phi\right)\Psi(\mathbf{x},t) = \left[\alpha\cdot(\mathbf{p}-q\mathbf{A}) + \beta m\right]\Psi(\mathbf{x},t). \tag{D.119}$$

This equation incorporates the charge $q$ with an external field. It is invariant by a gauge transformation:

$$A^\mu \to A^\mu - \partial^\mu\chi(x), \quad \Psi(x) \to e^{iq\chi(x)}\Psi(x). \tag{D.120}$$

We can remove the temporal dependence associated with a rest mass $m$, and splitting $\Psi$ in upper and lower components.

$$\Psi(\mathbf{x},t) = e^{-imt}\begin{pmatrix}\psi_U(\mathbf{x},t)\\ \psi_L(\mathbf{x},t)\end{pmatrix}. \tag{D.121}$$

In this equation, $\psi_U$ and $\psi_L$ are two-dimensional spinors (that is, $2\times 1$, or Pauli) still containing the temporal dependence of the kinetic energy. Replacing the above equation in the Dirac equation (D.119) we obtain that the terms with $\alpha$ couple $\psi_U$ and $\psi_L$, yielding

$$i\frac{\partial\psi_U}{\partial t} = \sigma\cdot(\mathbf{p}-q\mathbf{A})\,\psi_L + q\phi\psi_U, \tag{D.122}$$
$$i\frac{\partial\psi_L}{\partial t} = \sigma\cdot(\mathbf{p}-q\mathbf{A})\,\psi_U + q\phi\psi_L - 2m\psi_L. \tag{D.123}$$

## D.9 Pauli Equation

The assumption of time dependence (D.121) produces an asymmetric term proportional to $-2m$ in (D.123). This has two important consequences. The first is that in the nonrelativistic limit, $m\to\infty$, and $\partial\psi_L/\partial t$ becomes exceptionally large. This leads to the image of a charge jumping back and forth between the positive and negative energy components (known as *Zitterbewegung*). The second is that, if the kinetic energy is small compared to its rest mass, the lower components $\psi_L$ are small compared to the upper component. This is formally seen solving (D.123) for $\psi_L$:

$$\psi_L = \frac{\sigma \cdot (\mathbf{p} - q\mathbf{A})}{2m}\psi_U - \frac{i\,(\partial/\partial t) - q\phi}{2m}\psi_L. \tag{D.124}$$

Solving for $\psi_L$ and replacing it in (D.122) for $\psi_U$ we get the Klein-Gordon equation.

An approximation for the lower component is obtained expanding the equation above in a power series on ratios between momentum, energy, and the rest mass $m$:

$$\psi_L \simeq \psi_{L0}\left[O(v)^0\right] + \psi_{L1}\left[O(v)\right] + \cdots,$$

$$\psi_{L0} \simeq 0, \qquad \psi_{L1} \simeq \frac{\sigma \cdot (\mathbf{p} - q\mathbf{A})}{2m}\psi_U. \tag{D.125}$$

We can approximately decouple $\psi_L$ and $\psi_U$, inserting $\psi_{L1}$ in (D.120):

$$i\frac{\partial \psi_{U_0}}{\partial t} \simeq q\phi_{U_0} + \frac{\left[\sigma \cdot (\mathbf{p} - q\mathbf{A})\right]\left[\sigma \cdot (\mathbf{p} - q\mathbf{A})\right]}{2m}\psi_{U_0}. \tag{D.126}$$

Using the relation

$$\sigma \cdot \mathbf{A}\sigma \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B} + i\sigma \cdot (\mathbf{A} \times \mathbf{B}), \tag{D.127}$$

we can rewrite (D.126) as

$$i\frac{\partial \psi_{U_0}}{\partial t} \simeq \frac{(\mathbf{p} - q\mathbf{A})^2}{2m}\psi_{U_0} - \frac{q}{2m}\left[\sigma \cdot (\nabla \times \mathbf{A} + \mathbf{A} \times \nabla)\right]\psi_{U_0} + q\phi_{U_0}. \tag{D.128}$$

Since $\mathbf{p}$ is the operator $-i\nabla$ acting to the right, the terms $\nabla \times \mathbf{A}$ and $\mathbf{A} \times \nabla$ *do not* cancel. To calculate the gradient terms we use the vector identity

$$\begin{aligned}(\nabla \times \mathbf{A} + \mathbf{A} \times \nabla)\,\psi_U &= \nabla \times (\mathbf{A}\psi_U) + \mathbf{A} \times \nabla\psi_U \\ &= \psi_U \nabla \times \mathbf{A} + (\nabla\psi_U) \times \mathbf{A} + \mathbf{A} \times \nabla\psi_U \\ &= \psi_U\,(\nabla \times \mathbf{A}) = \psi_U \mathbf{B}.\end{aligned} \tag{D.129}$$

We thus obtain the *Pauli equation*,

$$i\frac{\partial \psi_{U_0}}{\partial t} \simeq H_P \psi_{U_0}, \tag{D.130}$$

which has the form of the Schrödinger equation, but with the Pauli Hamiltonian

$$H_P = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} - \frac{q}{2m}\sigma \cdot \mathbf{B} + q\phi. \tag{D.131}$$

Although the Dirac equation automatically includes higher order terms than the Pauli equation, we see that even in a lower approximation, the Dirac theory predicts a gyromagnetic factor $g = 2$ for fermions. Specifically, knowing that the magnetic dipole interaction with an external magnetic field is given by

$$H' = -\mu \cdot \mathbf{B}, \tag{D.132}$$

we can identify this with the second term of the Pauli Hamiltonian, eq. D.131, and we deduce the magnetic dipole moment $\mu$ and the $g$–factor as

$$\mu = -\mu_B g \mathbf{S} \equiv -\mu_B \sigma \qquad \left(\mu_B = \frac{q}{2m}\right)$$

$$\Rightarrow g = g_D = 2, \tag{D.133}$$

where $\mu_B$ is the Bohr magneton. Thus, the Dirac equation predicts that any particle with spin $\frac{1}{2}$ has $g = 2$.

The prediction (D.133) agrees perfectly with the experiments for electrons and muons in which case $q = e$ and $m$ is the particle mass (e.g., $m_\mu \simeq 205 m_e$). The small deviation from $g = 2$ for the electron, or the muon, is due to radiative corrections. In contrast, the magnetic moments of strongly interaction particles, like protons and neutrons, are very different from the Dirac predictions of 2 and 0 respectively:

$$\frac{g_p}{2} = 2.7928474, \quad \frac{g_n}{2} = -1.9130427. \tag{D.134}$$

The deviations from the Dirac value $\mu_D$ are characterized by the *anomalous momentum* $\kappa$,

$$\mu = \mu_D + \kappa \mu_B. \tag{D.135}$$

This anomalous magnetic interaction is included phenomenologically in the Dirac theory by the addition of an explicit term $-\kappa F_{\mu\nu}\sigma^{\mu\nu}/2$ to the Hamiltonian. The nucleon momenta can be explained, in principle, using the quark model.

### *D.9.1 Spin-orbit and Darwin terms*

Higher order corrections beyond the Pauli Hamiltonian can be obtained by replacing $\psi_{L1}$ for $\psi_L$ in (D.122). One gets

$$\Psi_{L_2} \simeq \frac{\sigma \cdot (\mathbf{p} - q\mathbf{A})}{2m}\psi_U - \frac{\left[i\left(\partial/\partial t - q\phi\right)\right]\left[\sigma \cdot (\mathbf{p} - q\mathbf{A})\right]}{2m}\psi_U. \tag{D.136}$$

The next step is to insert (D.136) in the equation for $\psi_U$, (D.122), and to identify the right side as $H\psi_U$. But, before that, the wave equation must be renormalized to make the Hamiltonian Hermitian and therefore produce a correct nonrelativistic limit. After resolving this complication, we get

$$\begin{aligned} H_0 \simeq & \left[\frac{(\mathbf{p} - q\mathbf{A})^2}{2m} - \frac{p^4}{8m^3}\right] + q\phi - \frac{q}{2m}\frac{\sigma \cdot \mathbf{B}}{2m} - \frac{iq}{8m^2}\mathbf{p} \cdot \mathbf{E} \\ & - \left[\frac{iq}{8m^2}\sigma \cdot (\nabla \times \mathbf{E}) + \frac{q}{4m^2}\sigma \cdot (\mathbf{E} \times \mathbf{p})\right], \end{aligned} \tag{D.137}$$

We recognize the second term inside the first parenthesis as a relativistic correction to the kinetic energy

$$\sqrt{p^2 + m^2} - m \simeq \frac{p^2}{2m} - \frac{p^4}{8m^3} + \cdots. \tag{D.138}$$

This is familiar in atomic physics, where it slightly lowers energy levels compared to the nonrelativistic values. The term $\sigma \cdot (\nabla \times \mathbf{E})$ in (D.137) is identically zero for a spherically symmetric potential. But the second term is zero for a spherically symmetric potential. But the term $\sigma \cdot (\mathbf{E} \times \mathbf{p})$ contains the *spin-orbit interaction* responsible for the *fine-structure* *effect* of the atomic levels. To show this we rewrite this term as

$$H_{so} = -\frac{q}{4m^2}\sigma\cdot(\mathbf{E}\times\mathbf{p}) = \frac{q}{4m^2}\sigma\cdot\frac{\partial V(r)}{\partial r}\frac{\mathbf{r}}{r}\times\mathbf{p}$$
$$= \frac{q}{4m^2}\frac{1}{r}\frac{\partial V(r)}{\partial r}\sigma\cdot\mathbf{l}. \qquad \text{(D.139)}$$

The term $\mathbf{p}\cdot\mathbf{E}$ in (D.137), known as the Darwin term, is related to the Laplacian of the central potential

$$V_{dar} = -\frac{iq}{8m^2}\mathbf{p}\cdot\mathbf{E} = \frac{1}{8m^2}\nabla\cdot\nabla V(r) = \frac{1}{8m^2}\nabla^2 V(r). \qquad \text{(D.140)}$$

Because $\nabla^2(1/r) \propto \delta(r)$, this is a contact interaction and thus only affects the $S$ states in atoms. Further insight on the nature of this term can be obtained by considering a charge that, when confined to a bound state, can oscillate between states of positive and negative energies To take this assumption further, imagine that this *Zitterbewegung* leads the charge to select a region in space of the size of its Compton wavelength $\Delta r \simeq 1/m$ around a point $r$. As a consequence, the Hamiltonian contains an extra term to account for this fluctuation:

$$H' \simeq \langle V(r+\Delta r)\rangle - \langle V(r)\rangle$$
$$\simeq \left\langle V(r) + \frac{\partial V}{\partial r}\Delta r + \frac{1}{2}\sum_{i,j}\Delta r_i \Delta r_j \frac{\partial^2 V}{\partial r_i \partial r_j}\right\rangle - \langle V(r)\rangle$$
$$\simeq \frac{1}{6}(\Delta r)^2\nabla^2 V \simeq \frac{1}{6m^2}\nabla^2 V. \qquad \text{(D.141)}$$

This indeed resembles (D.140).

# Appendix

The values inside pa
line the error in the e

## E.1 Constants

Electric charge

Planck constant

Speed of light

Gravitational constant

Boltzmann constant

Avogadro number

Molar volume

Faraday constant

Compton wavelength